ELIAS ZAKON

MATHEMATICAL ANALYSIS

2

The Zakon Series on Mathematical Analysis

Mathematical Analysis

Volume II

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The Trillia Group West Lafayette, IN

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Preface

This is a multipurpose text. When taken in full, including the "starred" sections, it is a graduate course covering differentiation on normed spaces and integration with respect to complex and vector-valued measures. The starred sections may be omitted without loss of continuity, however, for a junior or senior course. One also has the option of limiting all to E^n, or taking Riemann integration *before* Lebesgue theory (we call it the "limited approach"). The proofs and definitions are so chosen that they are as simple in the general case as in the more special cases. In a nutshell, the basic ideas of measure theory are given in Chapter 7, §§1 and 2. Not much more is needed for the "limited approach."

In Chapter 6 (Differentiation), we have endeavored to present a modern theory, without losing contact with the classical terminology and notation. (Otherwise, the student is unable to read classical texts after have been taught the "elegant" modern theory.) This is why we prefer to define derivatives as in classical analysis, i.e., as *numbers* or *vectors*, not as linear mappings. The latter are used to define a modern version of *differentials*.

In Chapter 9, we single out those calculus topics (e.g., improper integrals) that are best treated in the context of Lebesgue theory.

Our principle is to keep the exposition more general whenever the general case can be handled as simply as the special ones (the degree of the desired specialization is left to the instructor). Often this even simplifies matters—for example, by considering normed spaces instead of E^n only, one avoids cumbersome coordinate techniques. Doing so also makes the text more flexible.

Publisher's Notes

Text passages in blue are hyperlinks to other parts of the text.

Several annotations are used throughout this book:

- * This symbol marks material that can be omitted at first reading.
- ⇒ This symbol marks exercises that are of particular importance.

About the Author

Elias Zakon was born in Russia under the czar in 1908, and he was swept along in the turbulence of the great events of twentieth-century Europe.

Zakon studied mathematics and law in Germany and Poland, and later he joined his father's law practice in Poland. Fleeing the approach of the German Army in 1941, he took his family to Barnaul, Siberia, where, with the rest of the populace, they endured five years of hardship. The Leningrad Institute of Technology was also evacuated to Barnaul upon the siege of Leningrad, and there Zakon met the mathematician I. P. Natanson; with Natanson's encouragement, Zakon again took up his studies and research in mathematics.

Zakon and his family spent the years from 1946 to 1949 in a refugee camp in Salzburg, Austria, where he taught himself Hebrew, one of the six or seven languages in which he became fluent. In 1949, he took his family to the newly created state of Israel and he taught at the Technion in Haifa until 1956. In Israel he published his first research papers in logic and analysis.

Throughout his life, Zakon maintained a love of music, art, politics, history, law, and especially chess; it was in Israel that he achieved the rank of chess master.

In 1956, Zakon moved to Canada. As a research fellow at the University of Toronto, he worked with Abraham Robinson. In 1957, he joined the mathematics faculty at the University of Windsor, where the first degrees in the newly established Honours program in Mathematics were awarded in 1960. While at Windsor, he continued publishing his research results in logic and analysis. In this post-McCarthy era, he often had as his house-guest the prolific and eccentric mathematician Paul Erdős, who was then banned from the United States for his political views. Erdős would speak at the University of Windsor, where mathematicians from the University of Michigan and other American universities would gather to hear him and to discuss mathematics.

While at Windsor, Zakon developed three volumes on mathematical analysis, which were bound and distributed to students. His goal was to introduce rigorous material as early as possible; later courses could then rely on this material. We are publishing here the latest complete version of the last of these volumes, which was used in a two-semester class required of all Honours Mathematics students at Windsor.

Chapter 6

Differentiation on E^n and Other Normed Linear Spaces

§1. Directional and Partial Derivatives

In Chapter 5 we considered functions $f\colon E^1 \to E$ of one real variable.

Now we take up functions $f\colon E' \to E$ where *both* E' and E are *normed spaces*.[1]

The scalar field of both is always assumed *the same*: E^1 or C (the complex field). The case $E = E^*$ is excluded here; thus all is assumed *finite*.

We mostly use *arrowed* letters $\vec{p}, \vec{q}, \ldots, \vec{x}, \vec{y}, \vec{z}$ for vectors in the *domain space* E', and nonarrowed letters for those in E and for scalars.

As before, we adopt the convention that f is defined on *all* of E', with $f(\vec{x}) = 0$ if not defined otherwise.

Note that, if $\vec{p} \in E'$, one can express *any* point $\vec{x} \in E'$ as

$$\vec{x} = \vec{p} + t\vec{u},$$

with $t \in E^1$ and $\vec{u}$ a *unit* vector. For if $\vec{x} \neq \vec{p}$, set

$$t = |\vec{x} - \vec{p}| \text{ and } \vec{u} = \frac{1}{t}(\vec{x} - \vec{p});$$

and if $\vec{x} = \vec{p}$, set $t = 0$, and *any* $\vec{u}$ will do. We often use the notation

$$\vec{t} = \Delta\vec{x} = \vec{x} - \vec{p} = t\vec{u} \quad (t \in E^1,\ \vec{t}, \vec{u} \in E').$$

First of all, we generalize Definition 1 in Chapter 5, §1.

Definition 1.

Given $f\colon E' \to E$ and $\vec{p}, \vec{u} \in E'$ $(\vec{u} \neq \vec{0})$, we define the *directional derivative of f along $\vec{u}$* (or $\vec{u}$-directed derivative of f) at $\vec{p}$ by

$$D_{\vec{u}} f(\vec{p}) = \lim_{t \to 0} \frac{1}{t}[f(\vec{p} + t\vec{u}) - f(\vec{p})], \tag{1}$$

---

[1] We now presuppose §§9–12 of Chapter 3, including the "starred" parts.

if this limit exists in E (*finite*).

We also define the *$\vec{u}$-directed derived function*,

$$D_{\vec{u}}f\colon E' \to E,$$

as follows. For any $\vec{p} \in E'$,

$$D_{\vec{u}}f(\vec{p}) = \begin{cases} \lim_{t\to 0} \dfrac{1}{t}[f(\vec{p}+t\vec{u}) - f(\vec{p})] & \text{if this limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus $D_{\vec{u}}f$ is always defined, but the name *derivative* is used only if the limit (1) exists (finite). If it exists for each $\vec{p}$ in a set $B \subseteq E'$, we call $D_{\vec{u}}f$ (in classical notation $\partial f/\partial\vec{u}$) the *$\vec{u}$-directed derivative of f on B.*

Note that, as $t \to 0$, $\vec{x}$ tends to $\vec{p}$ over the line $\vec{x} = \vec{p} + t\vec{u}$. Thus $D_{\vec{u}}f(\vec{p})$ can be treated as a *relative* limit over that line. Observe that it depends on both the direction and the *length* of $\vec{u}$. Indeed, we have the following result.

Corollary 1. *Given $f\colon E' \to E$, $\vec{u} \neq \vec{0}$, and a scalar $s \neq 0$, we have*

$$D_{s\vec{u}}f = sD_{\vec{u}}f.$$

Moreover, $D_{s\vec{u}}f(\vec{p})$ is a genuine *derivative iff $D_{\vec{u}}f(\vec{p})$ is.*

Proof. Set $t = s\theta$ in (1) to get

$$sD_{\vec{u}}f(\vec{p}) = \lim_{\theta\to 0} \frac{1}{\theta}[f(\vec{p}+\theta s\vec{u}) - f(\vec{p})] = D_{s\vec{u}}f(\vec{p}). \quad \square$$

In particular, taking $s = 1/|\vec{u}|$, we have

$$|s\vec{u}| = \frac{|\vec{u}|}{|\vec{u}|} = 1 \text{ and } D_{\vec{u}}f = \frac{1}{s}D_{s\vec{u}}f.$$

Thus all reduces to the case $D_{\vec{v}}f$, where $\vec{v} = s\vec{u}$ is a *unit* vector. This device, called *normalization*, is often used, but actually it does not simplify matters.

If $E' = E^n$ (C^n), then f is a function of n scalar variables x_k ($k = 1, \ldots, n$) and E' has the n basic unit vectors $\vec{e}_k$. This example leads us to the following definition.

Definition 2.

If in formula (1), $E' = E^n$ (C^n) and $\vec{u} = \vec{e}_k$ for a fixed $k \leq n$, we call $D_{\vec{u}}f$ the *partially derived function for f, with respect to x_k,* denoted

$$D_kf \text{ or } \frac{\partial f}{\partial x_k},$$

and the limit (1) is called the *partial derivative of f at $\vec{p}$, with respect to* x_k, denoted

$$D_k f(\vec{p}), \text{ or } \frac{\partial}{\partial x_k} f(\vec{p}), \text{ or } \left.\frac{\partial f}{\partial x_k}\right|_{\vec{x}=\vec{p}}.$$

If it exists for all $\vec{p} \in B$, we call $D_k f$ the *partial derivative* (briefly, *partial*) *of f on B*, with respect to x_k.

In any case, the derived *functions* $D_k f$ $(k = 1, \ldots, n)$ are *always* defined on all of E^n (C^n).

If $E' = E^3$ (C^3), we often write x, y, z for x_1, x_2, x_3, and

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \text{ for } D_k f \quad (k = 1, 2, 3).^2$$

Note 1. If $E' = E^1$, scalars are also "vectors," and $D_1 f$ coincides with f' as defined in Chapter 5, §1 (except where $f' = \pm\infty$). Explain!

Note 2. As we have observed, the $\vec{u}$-directed derivative (1) is obtained by keeping $\vec{x}$ *on the line* $\vec{x} = \vec{p} + t\vec{u}$.

If $\vec{u} = \vec{e}_k$, the line is parallel to the kth axis; so all coordinates of $\vec{x}$, except x_k, remain *fixed* $(x_i = p_i,\ i \neq k)$, and f behaves like a function of *one* variable, x_k. Thus we can compute $D_k f$ by the usual rules of differentiation, treating all x_i $(i \neq k)$ as *constants* and x_k as the only variable.

For example, let $f(x, y) = x^2 y$. Then

$$\frac{\partial f}{\partial x} = D_1 f(x, y) = 2xy \text{ and } \frac{\partial f}{\partial y} = D_2 f(x, y) = x^2.$$

Note 3. More generally, given $\vec{p}$ and $\vec{u} \neq \vec{0}$, set

$$h(t) = f(\vec{p} + t\vec{u}), \quad t \in E^1.$$

Then $h(0) = f(\vec{p})$; so

$$\begin{aligned} D_{\vec{u}} f(\vec{p}) &= \lim_{t \to 0} \frac{1}{t}[f(\vec{p} + t\vec{u}) - f(\vec{p})] \\ &= \lim_{t \to 0} \frac{h(t) - h(0)}{t - 0} \\ &= h'(0) \end{aligned}$$

if the limit exists. Thus all reduces to a function h of *one* real variable.

For functions $f\colon E^1 \to E$, the existence of a finite derivative ("differentiability") at p implies continuity at p (Theorem 1 of Chapter 5, §1). But in the general case, $f\colon E' \to E$, *this may fail even if $D_{\vec{u}} f(\vec{p})$ exists for all $\vec{u} \neq \vec{0}$.*

---

[2] Similarly in the case $E' = E^2$ (C^2).

Examples.

(a) Define $f\colon E^2 \to E^1$ by

$$f(x,y) = \frac{x^2y}{x^4+y^2}, \quad f(0,0)=0.$$

Fix a unit vector $\vec{u} = (u_1, u_2)$ in E^2. Let $\vec{p} = (0,0)$. To find $D_{\vec{u}}f(p)$, use the h of Note 3:

$$h(t) = f(\vec{p} + t\vec{u}) = f(t\vec{u}) = f(tu_1, tu_2) = \frac{tu_1^2u_2}{t^2u_1^4 + u_2^2} \text{ if } u_2 \neq 0,$$

and $h = 0$ if $u_2 = 0$. Hence

$$D_{\vec{u}}f(\vec{p}) = h'(0) = \frac{u_1^2}{u_2} \text{ if } u_2 \neq 0,$$

and $h'(0) = 0$ if $u_2 = 0$. Thus $D_{\vec{u}}(\vec{0})$ exists for all $\vec{u}$. Yet f is discontinuous at $\vec{0}$ (see Problem 9 in Chapter 4, §3).

(b) Let

$$f(x,y) = \begin{cases} x+y & \text{if } xy = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Then $f(x,y) = x$ on the x-axis; so $D_1f(0,0) = 1$.

Similarly, $D_2f(0,0) = 1$. Thus both partials exist at $\vec{0}$.

Yet f is discontinuous at $\vec{0}$ (even relatively so) over any line $y = ax$ $(a \neq 0)$. For on that line, $f(x,y) = 1$ if $(x,y) \neq (0,0)$; so $f(x,y) \to 1$; but $f(0,0) = 0 + 0 = 0$.

Thus continuity at $\vec{0}$ *fails*. (But see Theorem 1 below!)

Hence, if differentiability *is* to imply continuity, it must be defined in a stronger manner. We do it in §3. For now, we prove only some theorems on partial and directional derivatives, based on those of Chapter 5.

Theorem 1. *If $f\colon E' \to E$ has a $\vec{u}$-directed derivative at $\vec{p} \in E'$, then f is* relatively *continuous at $\vec{p}$ over the line*

$$\vec{x} = \vec{p} + t\vec{u} \quad (\vec{0} \neq \vec{u} \in E').$$

Proof. Set $h(t) = f(\vec{p} + t\vec{u})$, $t \in E^1$.

By Note 3, our assumption implies that h (a function *on* E^1) is differentiable at 0.

By Theorem 1 in Chapter 5, §1, then, h is continuous at 0; so

$$\lim_{t\to 0} h(t) = h(0) = f(\vec{p}),$$

i.e.,

$$\lim_{t\to 0} f(\vec{p}+t\vec{u}) = f(\vec{p}).$$

But this means that $f(\vec{x}) \to f(\vec{p})$ as $\vec{x} \to \vec{p}$ *over the line* $\vec{x} = \vec{p}+t\vec{u}$, for, on that line, $\vec{x} = \vec{p}+t\vec{u}$.

Thus, indeed, f is relatively continuous at $\vec{p}$, as stated. □

Note that we actually used the substitution $\vec{x} = \vec{p}+t\vec{u}$. This is admissible since the dependence between x and t is one-to-tone (Corollary 2(iii) of Chapter 4, §2). Why?

Theorem 2. *Let* $E' \ni \vec{u} = \vec{q} - \vec{p}$, $\vec{u} \neq \vec{0}$.

If $f\colon E' \to E$ *is relatively continuous on the segment* $I = L[\vec{p}, \vec{q}]$ *and has a* $\vec{u}$*-directed derivative on* $I - Q$ *(*Q *countable), then*

$$|f(\vec{q}) - f(\vec{p})| \le \sup |D_{\vec{u}} f(\vec{x})|, \quad \vec{x} \in I - Q. \tag{2}$$

Proof. Set again $h(t) = f(\vec{p}+t\vec{u})$ and $g(t) = \vec{p}+t\vec{u}$.

Then $h = f \circ g$, and g is continuous on E^1. (Why?)

As f is relatively continuous on $I = L[\vec{p}, \vec{q}]$, so is $h = f \circ g$ on the interval $J = [0, 1] \subset E^1$ (cf. Chapter 4, §8, Example (1)).

Now fix $t_0 \in J$. If $\vec{x}_0 = \vec{p} + t_0\vec{u} \in I - Q$, our assumptions imply the existence of

$$\begin{aligned} D_{\vec{u}} f(\vec{x}_0) &= \lim_{t\to 0} \frac{1}{t}[f(\vec{x}_0 + t\vec{u}) - f(\vec{x}_0)] \\ &= \lim_{t\to 0} \frac{1}{t}[f(\vec{p}+t_0\vec{u}+t\vec{u}) - f(\vec{p}+t_0\vec{u})] \\ &= \lim_{t\to 0} \frac{1}{t}[h(t_0+t) - h(t_0)] \\ &= h'(t_0). \quad \text{(Explain!)} \end{aligned}$$

This can fail for at most a *countable* set Q' of points $t_0 \in J$ (those for which $\vec{x}_0 \in Q$).

Thus h is differentiable on $J - Q'$; and so, by Corollary 1 in Chapter 5, §4,

$$|h(1) - h(0)| \le \sup_{t \in J - Q'} |h'(t)| = \sup_{\vec{x} \in I - Q} |D_{\vec{u}} f(\vec{x})|.$$

Now as $h(1) = f(\vec{p}+\vec{u}) = f(\vec{q})$ and $h(\vec{0}) = f(\vec{p})$, formula (2) follows. □

Theorem 3. *If in Theorem 2,* $E = E^1$ *and if* f *has a* $\vec{u}$*-directed derivative at least on the* open *line segment* $L(\vec{p}, \vec{q})$*, then*

$$f(\vec{q}) - f(\vec{p}) = D_{\vec{u}} f(\vec{x}_0) \tag{3}$$

for some $\vec{x}_0 \in L(\vec{p}, \vec{q})$.

The proof is as in Theorem 2, based on Corollary 3 in Chapter 5, §2 (instead of Corollary 1 in Chapter 5, §4).

Theorems 2 and 3 are often used in "normalized" form, as follows.

Corollary 2. *If in Theorems* 2 *and* 3, *we set*

$$r = |\vec{u}| = |\vec{q} - \vec{p}| \neq 0 \text{ and } \vec{v} = \frac{1}{r}\vec{u},$$

then formulas (2) *and* (3) *can be written as*

$$|f(\vec{q}) - f(\vec{p})| \le |\vec{q} - \vec{p}| \sup |D_{\vec{v}} f(\vec{x})|, \quad \vec{x} \in I - Q, \tag{2'}$$

and

$$f(\vec{q}) - f(\vec{p}) = |\vec{q} - \vec{p}|\, D_{\vec{v}} f(\vec{x}_0) \tag{3'}$$

for some $\vec{x}_0 \in L(\vec{p}, \vec{q})$.

For by Corollary 1,

$$D_{\vec{u}} f = r D_{\vec{v}} f = |\vec{q} - \vec{p}|\, D_{\vec{v}} f;$$

so $(2')$ and $(3')$ follow.

Problems on Directional and Partial Derivatives

1. Complete all missing details in the proof of Theorems 1 to 3 and Corollaries 1 and 2.

2. Complete all details in Examples (a) and (b). Find $D_1 f(\vec{p})$ and $D_2 f(\vec{p})$ also for $\vec{p} \neq 0$. Do Example (b) in two ways: (i) use Note 3; (ii) use Definition 2 only.

3. In Examples (a) and (b) describe $D_{\vec{u}} f \colon E^2 \to E^1$. Compute it for $\vec{u} = (1, 1) = \vec{p}$.

In (b), show that f has no directional derivatives $D_{\vec{u}} f(\vec{p})$ except if $\vec{u} \parallel \vec{e}_1$ or $\vec{u} \parallel \vec{e}_2$. Give two proofs: (i) use Theorem 1; (ii) use definitions only.

4. Prove that if $f \colon E^n\,(C^n) \to E$ has a zero partial derivative, $D_k f = 0$, on a convex set A, then $f(\vec{x})$ does not depend on x_k, for $\vec{x} \in A$. (Use Theorems 1 and 2.)

5. Describe $D_1 f$ and $D_2 f$ on the various parts of E^2, and discuss the relative continuity of f over lines through $\vec{0}$, given that $f(x, y)$ equals:

(i) $\dfrac{xy}{x^2 + y^2}$; (ii) the integral part of $x + y$;

(iii) $\dfrac{xy}{|x|} + x \sin \dfrac{1}{y}$; (iv) $xy \dfrac{x^2 - y^2}{x^2 + y^2}$;

(v) $\sin(y \cos x)$; (vi) x^y.

(Set $f = 0$ wherever the formula makes no sense.)

⇒**6.** Prove that if $f\colon E' \to E^1$ has a local maximum or minimum at $\vec{p} \in E'$, then $D_{\vec{u}} f(\vec{p}) = 0$ for every vector $\vec{u} \neq \vec{0}$ in E'.
[Hint: Use Note 3, then Corollary 1 in Chapter 5, §2.]

7. State and prove the Finite Increments Law (Theorem 1 of Chapter 5, §4) for *directional* derivatives.
[Hint: Imitate Theorem 2 using *two* auxiliary functions, h and k.]

8. State and prove Theorems 4 and 5 of Chapter 5, §1, for *directional* derivatives.

§2. Linear Maps and Functionals. Matrices

For an adequate definition of differentiability, we need the notion of a *linear map*. Below, E', E'', and E denote normed spaces over *the same* scalar field, E^1 or C.

Definition 1.

A function $f\colon E' \to E$ is a *linear map* if and only if for all $\vec{x}, \vec{y} \in E'$ and scalars a, b

$$f(a\vec{x} + b\vec{y}) = af(\vec{x}) + bf(\vec{y}); \tag{1}$$

equivalently, iff for all such $\vec{x}$, $\vec{y}$, and a

$$f(\vec{x} + \vec{y}) = f(x) + f(y) \text{ and } f(a\vec{x}) = af(\vec{x}). \text{ (Verify!)}$$

If $E = E'$, such a map is also called a *linear operator*.

If the range space E is the *scalar field of* E', (i.e., E^1 or C,) the linear map f is also called a (real or complex) *linear functional* on E'.

Note 1. Induction extends formula (1) to any "linear combinations":

$$f\left(\sum_{i=1}^{m} a_i \vec{x}_i\right) = \sum_{i=1}^{m} a_i f(\vec{x}_i) \tag{2}$$

for all $\vec{x}_i \in E'$ and scalars a_i.

Briefly: *A linear map f preserves linear combinations.*

Note 2. Taking $a = b = 0$ in (1), we obtain $f(\vec{0}) = 0$ *if f is linear*.

Examples.

(a) Let $E' = E^n$ (C^n). Fix a vector $\vec{v} = (v_1, \ldots, v_n)$ in E' and set

$$(\forall\, \vec{x} \in E') \quad f(\vec{x}) = \vec{x} \cdot \vec{v}$$

(*inner product*; see Chapter 3, §§1–3 and §9).

Then

$$\begin{aligned} f(a\vec{x}+b\vec{y}) &= (a\vec{x})\cdot\vec{v}+(b\vec{y})\cdot\vec{v} \\ &= a(\vec{x}\cdot\vec{v})+b(\vec{y}\cdot\vec{v}) \\ &= af(\vec{x})+bf(\vec{y}); \end{aligned}$$

so f *is linear*. Note that if $E' = E^n$, then by definition,

$$f(\vec{x}) = \vec{x}\cdot\vec{v} = \sum_{k=1}^{n} x_k v_k = \sum_{k=1}^{n} v_k x_k.$$

If, however, $E' = C^n$, then

$$f(\vec{x}) = \vec{x}\cdot\vec{v} = \sum_{k=1}^{n} x_k \bar{v}_k = \sum_{k=1}^{n} \bar{v}_k x_k,$$

where $\bar{v}_k$ is the *conjugate* of the complex number v_k.

By Theorem 3 in Chapter 4, §3, f is continuous (a polynomial!).

Moreover, $f(\vec{x}) = \vec{x}\cdot\vec{v}$ is a *scalar* (in E^1 or C). Thus the range of f lies in the *scalar field* of E'; so f is a *linear functional* on E'.

(b) Let $I = [0, 1]$. Let E' be the set of all functions $u\colon I \to E$ that are *of class* CD^∞ (Chapter 5, §6) on I, hence bounded there (Theorem 2 of Chapter 4, §8).

As in Example (C) in Chapter 3, §10, E' is a normed linear space, with norm

$$\|u\| = \sup_{x\in I} |u(x)|.$$

Here each function $u \in E'$ is treated as a single "*point*" in E'. The distance between two such points, u and v, equals $\|u - v\|$, by definition.

Now define a map D on E' by setting $D(u) = u$ (derivative of u on I). As every $u \in E'$ is of class CD^∞, so is u'.

Thus $D(u) = u' \in E'$, and so $D\colon E' \to E'$ is a *linear operator*. (Its linearity follows from Theorem 4 in Chapter 5, §1.)

(c) Let again $I = [0, 1]$. Let E' be the set of all functions $u\colon I \to E$ that are bounded and have *antiderivatives* (Chapter 5, §5) on I. With norm $\|u\|$ as in Example (b), E' is a normed linear space.

Now define $\phi\colon E' \to E$ by

$$\phi(u) = \int_0^1 u,$$

with $\int u$ as in Chapter 5, §5. (Recall that $\int_0^1 u$ is *an element of* E if $u\colon I \to E$.) By Corollary 1 in Chapter 5, §5, ϕ is a linear map of E' into

E. (Why?)

(d) The *zero map* $f = 0$ on E' is always linear. (Why?)

Theorem 1. *A linear map* $f\colon E' \to E$ *is continuous* (*even uniformly so*) *on* all *of* E' *iff it is continuous at* $\vec{0}$; *equivalently, iff there is a real* $c > 0$ *such that*

$$(\forall\, \vec{x} \in E') \quad |f(\vec{x})| \le c|\vec{x}|.$$

(*We call this property* linear boundedness.)

Proof. Assume that f is continuous at $\vec{0}$. Then, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$|f(\vec{x}) - f(\vec{0})| = |f(\vec{x})| \le \varepsilon$$

whenever $|\vec{x} - \vec{0}| = |\vec{x}| < \delta$.

Now, for *any* $\vec{x} \ne \vec{0}$, we surely have

$$\left| \frac{\delta \vec{x}}{2|\vec{x}|} \right| = \frac{\delta}{2} < \delta.$$

Hence

$$(\forall\, \vec{x} \ne \vec{0}) \quad \left| f\left(\frac{\delta \vec{x}}{2|\vec{x}|} \right) \right| \le \varepsilon,$$

or, by linearity,

$$\frac{\delta}{2|\vec{x}|} |f(\vec{x})| \le \varepsilon,$$

i.e.,

$$|f(\vec{x})| \le \frac{2\varepsilon}{\delta} |\vec{x}|.$$

By Note 2, this also holds if $\vec{x} = \vec{0}$.

Thus, taking $c = 2\varepsilon/\delta$, we obtain

$$(\forall\, \vec{x} \in E') \quad f(\vec{x}) \le c|\vec{x}| \quad \text{(linear boundedness)}. \tag{3}$$

Now assume (3). Then

$$(\forall\, \vec{x}, \vec{y} \in E') \quad |f(\vec{x} - \vec{y})| \le c|\vec{x} - \vec{y}|;$$

or, by linearity,

$$(\forall\, \vec{x}, \vec{y} \in E') \quad |f(\vec{x}) - f(\vec{y})| \le c|\vec{x} - \vec{y}|.^{1} \tag{4}$$

Hence f is uniformly continuous (given $\varepsilon > 0$, take $\delta = \varepsilon/c$). This, in turn, implies continuity at $\vec{0}$; so all conditions are equivalent, as claimed. □

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[1] This is the so-called *uniform Lipschitz condition.*

A linear map *need not* be continuous.[2] But, for E^n and C^n, we have the following result.

Theorem 2.

(i) *Any linear map on E^n or C^n is uniformly continuous.*

(ii) *Every linear* functional *on E^n (C^n) has the form*

$$f(\vec{x}) = \vec{x} \cdot \vec{v} \quad (\textit{dot product})$$

for some unique vector $\vec{v} \in E^n$ (C^n), dependent on f only.

Proof. Suppose $f\colon E^n \to E$ is linear; so f preserves linear combinations.

But *every* $\vec{x} \in E^n$ is such a combination,

$$\vec{x} = \sum_{k=1}^{n} x_k \vec{e}_k \quad \text{(Theorem 2 in Chapter 3, §§1–3).}$$

Thus, by Note 1,

$$f(\vec{x}) = f\left(\sum_{k=1}^{n} x_k \vec{e}_k\right) = \sum_{k=1}^{n} x_k f(\vec{e}_k).$$

Here the function values $f(\vec{e}_k)$ are fixed vectors in the range space E, say,

$$f(\vec{e}_k) = v_k \in E,$$

so that

$$f(\vec{x}) = \sum_{k=1}^{n} x_k f(\vec{e}_k) = \sum_{k=1}^{n} x_k v_k, \quad v_k \in E. \tag{5}$$

Thus f is a *polynomial* in n real variables x_k, hence continuous (even uniformly so, by Theorem 1).

In particular, if $E = E^1$ (i.e., f is a linear *functional*) then all v_k in (5) are real numbers; so they form a *vector*

$$\vec{v} = (v_1, \ldots, v_k) \text{ in } E^n,$$

and (5) can be written as

$$f(\vec{x}) = \vec{x} \cdot \vec{v}.$$

The vector $\vec{v}$ is unique. For suppose there are *two* vectors, $\vec{u}$ and $\vec{v}$, such that

$$(\forall\, \vec{x} \in E^n) \quad f(\vec{x}) = \vec{x} \cdot \vec{v} = \vec{x} \cdot \vec{u}.$$

Then

$$(\forall\, \vec{x} \in E^n) \quad \vec{x} \cdot (\vec{v} - \vec{u}) = 0.$$

---

[2] See Problem 2(ii) below.

By Problem 10 of Chapter 3, §§1–3, this yields $\vec{v} - \vec{u} = \vec{0}$, or $\vec{v} = \vec{u}$. This completes the proof for $E = E^n$.

It is analogous for C^n; only in (ii) the v_k are *complex* and one has to replace them by their *conjugates* $\bar{v}_k$ when forming the vector $\vec{v}$ to obtain $f(\vec{x}) = \vec{x} \cdot \vec{v}$. Thus all is proved. □

Note 3. Formula (5) shows that a linear map $f\colon E^n\,(C^n) \to E$ is *uniquely determined* by the n function values $v_k = f(\vec{e}_k)$.

If further $E = E^m$ (C^m), the vectors v_k are *m-tuples* of scalars,

$$v_k = (v_{1k}, \ldots, v_{mk}).$$

We often write such vectors *vertically*, as the n "columns" in an array of m "rows" and n "columns":

$$\begin{pmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ v_{21} & v_{22} & \ldots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{m1} & v_{m2} & \ldots & v_{mn} \end{pmatrix}. \tag{6}$$

Formally, (6) is a double sequence of mn terms, called an *$m \times n$ matrix.* We denote it by $[f] = (v_{ik})$, where for $k = 1, 2, \ldots, n$,

$$f(\vec{e}_k) = v_k = (v_{1k}, \ldots, v_{mk}).$$

Thus *linear maps $f\colon E^n \to E^m$ (or $f\colon C^n \to C^m$) correspond one-to-one to their matrices $[f]$.*

The easy proof of Corollaries 1 to 3 below is left to the reader.

Corollary 1. *If $f, g\colon E' \to E$ are linear, so is*

$$h = af + bg$$

for any scalars a, b.

If further $E' = E^n$ (C^n) and $E = E^m$ (C^m), with $[f] = (v_{ik})$ and $[g] = (w_{ik})$, then

$$[h] = (av_{ik} + bw_{ik}).$$

Corollary 2. *A map $f\colon E^n\,(C^n) \to E$ is linear iff*

$$f(\vec{x}) = \sum_{k=1}^{n} v_k x_k,$$

where $v_k = f(\vec{e}_k)$.

Hint: For the "if," use Corollary 1. For the "only if," use formula (5) above.

Corollary 3. *If* $f\colon E'\to E''$ *and* $g\colon E''\to E$ *are linear, so is the composite* $h=g\circ f$.

Our next theorem deals with the *matrix* of the composite linear map $g\circ f$.

Theorem 3. *Let* $f\colon E'\to E''$ *and* $g\colon E''\to E$ *be linear, with*

$$E'=E^n\ (C^n),\ E''=E^m\ (C^m),\ \text{and } E=E^r\ (C^r).$$

If $[f]=(v_{ik})$ *and* $[g]=(w_{ji})$, *then*

$$[h]=[g\circ f]=(z_{jk}),$$

where

$$z_{jk}=\sum_{i=1}^{m} w_{ji}v_{ik},\quad j=1,2,\ldots,r,\ k=1,2,\ldots,n. \tag{7}$$

Proof. Denote the basic unit vectors in E' by

$$e'_1,\ldots,e'_n,$$

those in E'' by

$$e''_1,\ldots,e''_m,$$

and those in E by

$$e_1,\ldots,e_r.$$

Then for $k=1,2,\ldots,n$,

$$f(e'_k)=v_k=\sum_{i=1}^{m} v_{ik}e''_i \text{ and } h(e'_k)=\sum_{j=1}^{r} z_{jk}e_j,$$

and for $i=1,\ldots\ m$,

$$g(e''_i)=\sum_{j=1}^{r} w_{ji}e_j.$$

Also,

$$h(e'_k)=g(f(e'_k))=g\left(\sum_{i=1}^{m} v_{ik}e''_i\right)=\sum_{i=1}^{m} v_{ik}g(e''_i)=\sum_{i=1}^{m} v_{ik}\left(\sum_{j=1}^{r} w_{ji}e_j\right).$$

Thus

$$h(e'_k)=\sum_{j=1}^{r} z_{jk}e_j=\sum_{j=1}^{r}\left(\sum_{i=1}^{m} w_{ji}v_{ik}\right)e_j.$$

But the representation in terms of the e_j is *unique* (Theorem 2 in Chapter 3, §§1–3), so, equating coefficients, we get (7). □

Note 4. Observe that z_{jk} is obtained, so to say, by "dot-multiplying" the jth *row* of $[g]$ (an $r \times m$ matrix) by the kth *column* of $[f]$ (an $m \times n$ matrix).

It is natural to set

$$[g]\,[f] = [g \circ f],$$

or

$$(w_{ji})(v_{ik}) = (z_{jk}),$$

with z_{jk} as in (7).

Caution. Matrix multiplication, so defined, is *not commutative.*

Definition 2.

The set of all *continuous* linear maps $f\colon E' \to E$ (for fixed E' and E) is denoted $L(E', E)$.

If $E = E'$, we write $L(E)$ instead.

For each f in $L(E', E)$, we define its *norm* by

$$\|f\| = \sup_{|\vec{x}| \leq 1} |f(\vec{x})|.^{3}$$

Note that $\|f\| < +\infty$, by Theorem 1.

Theorem 4. *$L(E', E)$ is a normed linear space under the norm defined above and under the usual operations on functions, as in Corollary* 1.

Proof. Corollary 1 easily implies that $L(E', E)$ is a vector space. We now show that $\|\cdot\|$ is a genuine *norm.*

The triangle law,

$$\|f + g\| \leq \|f\| + \|g\|,$$

follows exactly as in Example (C) of Chapter 3, §10. (Verify!)

Also, by Problem 5 in Chapter 2, §§8–9, $\sup|af(\vec{x})| = |a| \sup|f(\vec{x})|$. Hence $\|af\| = |a|\|f\|$ for any scalar a.

As noted above, $0 \leq \|f\| < +\infty$.

It remains to show that $\|f\| = 0$ iff f is the zero map. If

$$\|f\| = \sup_{|\vec{x}| \leq 1} |f(\vec{x})| = 0,$$

then $|f(\vec{x})| = 0$ when $|\vec{x}| \leq 1$. Hence, if $\vec{x} \neq \vec{0}$,

$$f\Big(\frac{\vec{x}}{|\vec{x}|}\Big) = \frac{1}{|\vec{x}|} f(\vec{x}) = 0.$$

As $f(\vec{0}) = 0$, we have $f(\vec{x}) = 0$ for *all* $\vec{x} \in E'$.

Thus $\|f\| = 0$ implies $f = 0$, and the converse is clear. Thus all is proved. $\square$

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[3] Equivalently, $\|f\| = \sup_{\vec{x} \neq \vec{0}} |f(\vec{x})|/|\vec{x}|$; see Note 5 below.

Note 5. A similar proof, via $f\left(\frac{\vec{x}}{|\vec{x}|}\right)$ and properties of lub, shows that

$$\|f\| = \sup_{\vec{x}\neq 0}\left|\frac{f(\vec{x})}{|\vec{x}|}\right|$$

and

$$(\forall\, \vec{x} \in E') \quad |f(\vec{x})| \le \|f\|\,|\vec{x}|.$$

It also follows that $\|f\|$ is the *least* real c such that

$$(\forall\, \vec{x} \in E') \quad |f(\vec{x})| \le c|\vec{x}|.$$

Verify. (See Problem 3′.)

As in any normed space, we define distances in $L(E', E)$ by

$$\rho(f, g) = \|f - g\|,$$

making it a *metric* space; so we may speak of convergence, limits, etc., in it.

Corollary 4. *If $f \in L(E', E'')$ and $g \in L(E'', E)$, then*

$$\|g \circ f\| \le \|g\|\,\|f\|.$$

Proof. By Note 5,

$$(\forall\, \vec{x} \in E') \quad |g(f(\vec{x}))| \le \|g\|\,|f(\vec{x})| \le \|g\|\,\|f\|\,|\vec{x}|.$$

Hence

$$(\forall\, \vec{x} \neq \vec{0}) \quad \left|\frac{(g\circ f)(\vec{x})}{|\vec{x}|}\right| \le \|g\|\,\|f\|,$$

and so

$$\|g\|\,\|f\| \ge \sup_{\vec{x}\neq\vec{0}} \frac{|(g\circ f)(\vec{x})|}{|\vec{x}|} = \|g \circ f\|. \quad \square$$

Problems on Linear Maps and Matrices

1. Verify Note 1 and the equivalence of the two statements in Definition 1.

2. In Examples (b) and (c) show that

$$f_n \to f \ (\textit{uniformly}) \text{ on } I \text{ iff } \|f_n - f\| \to 0,$$

i.e., *$f_n \to f$ in E'.*

[Hint: Use Theorem 1 in Chapter 4, §2.]

Hence deduce the following.

(i) If E is complete, then the map ϕ in Example (c) is continuous.
[Hint: Use Theorem 2 of Chapter 5, §9, and Theorem 1 in Chapter 4, §12.]

(ii) The map D of Example (b) is not continuous.
[Hint: Use Problem 3 in Chapter 5, §9.]

3. Prove Corollaries 1 to 3.

3′. Show that

$$\|f\| = \sup_{|\vec{x}|\le 1} |f(\vec{x})| = \sup_{|\vec{x}|=1} |f(\vec{x})| = \sup_{\vec{x}\ne\vec{0}} \frac{|f(\vec{x})|}{|\vec{x}|}.$$

[Hint: From linearity of f deduce that $|f(\vec{x})| \ge |f(cx)|$ if $|c| < 1$. Hence one may disregard vectors of length < 1 when computing $\sup|f(\vec{x})|$. Why?]

4. Find the matrices $[f]$, $[g]$, $[h]$, $[k]$, and the defining formulas for the linear maps $f\colon E^2 \to E^1$, $g\colon E^3 \to E^4$, $h\colon E^4 \to E^2$, $k\colon E^1 \to E^3$ if

(i) $f(\vec{e}_1) = 3$, $f(\vec{e}_2) = -2$;

(ii) $g(\vec{e}_1) = (1, 0, -2, 4)$, $g(\vec{e}_2) = (0, 2, -1, 1)$, $g(\vec{e}_3) = (0, 1, 0, -1)$;

(iii) $h(\vec{e}_1) = (2, 2)$, $h(\vec{e}_2) = (0, -2)$, $h(\vec{e}_3) = (1, 0)$, $h(\vec{e}_4) = (-1, 1)$;

(iv) $k(1) = (0, 1, -1)$.

5. In Problem 4, use Note 4 to find the product matrices $[k]\,[f]$, $[g]\,[k]$, $[f]\,[h]$, and $[h]\,[g]$. Hence obtain the defining formulas for $k \circ f$, $g \circ k$, $f \circ h$, and $h \circ g$.

6. For $m \times n$-matrices (with m and n fixed) define addition and multiplication by scalars as follows:

$$a[f] + b[g] = [af + bg] \text{ if } f, g \in L(E^n, E^m) \text{ (or } L(C^n, C^m)).$$

Show that these matrices form a vector space over E^1 (or C).

7. With matrix addition as in Problem 6, and multiplication as in Note 4, show that all $n \times n$-matrices form a *noncommutative ring with unity*, i.e., satisfy the field axioms (Chapter 2, §§1–4) except the commutativity of multiplication and existence of multiplicative inverses (give counterexamples!).

Which is the "unity" matrix?

8. Let $f\colon E' \to E$ be linear. Prove the following statements.

(i) The derivative $D_{\vec{u}}f(\vec{p})$ exists and equals $f(\vec{u})$ for every $\vec{p}, \vec{u} \in E'$ $(\vec{u} \ne \vec{0})$;

(ii) f is relatively continuous on any *line* in E' (use Theorem 1 in §1);

(iii) f carries any such line into a line in E.

9. Let $g\colon E'' \to E$ be linear. Prove that if some $f\colon E' \to E''$ has a $\vec{u}$-directed derivative at $\vec{p} \in E'$, so has $h = g \circ f$, and $D_{\vec{u}}h(\vec{p}) = g(D_{\vec{u}}f(\vec{p}))$. [Hint: Use Problem 8.]

10. A set A in a vector space V ($A \subseteq V$) is said to be *linear* (or a *linear subspace* of V) iff $a\vec{x} + b\vec{y} \in A$ for any $\vec{x}, \vec{y} \in A$ and any scalars a, b. Prove the following.

(i) Any such A is itself a vector space.

(ii) If $f\colon E' \to E$ is a linear map and A is linear in E' (respectively, in E), so is $f[A]$ in E (respectively, so is $f^{-1}[A]$ in E').

11. A set A in a vector space V is called the *span* of a set $B \subseteq A$ ($A = \mathrm{sp}(B)$) iff A consists of all linear combinations of vectors from B. We then also say that B *spans* A.

Prove the following:

(i) $A = \mathrm{sp}(B)$ is the smallest linear subspace of V that contains B.

(ii) If $f\colon V \to E$ is linear and $A = \mathrm{sp}(B)$, then $f[A] = \mathrm{sp}(f[B])$ in E.

12. A set $B = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ in a vector space V is called a *basis* iff each $\vec{v} \in V$ has a *unique* representation as

$$\vec{v} = \sum_{i=1}^{n} a_i \vec{x}_i$$

for some scalars a_i. If so, the number n of the vectors in B is called the *dimension* of V, and V is said to be n*-dimensional*. Examples of such spaces are E^n and C^n (the $\vec{e}_k$ form a basis!).

(i) Show that B is a basis iff it *spans* V (see Problem 11) and its elements $\vec{x}_i$ are *linearly independent*, i.e.,

$$\sum_{i=1}^{n} a_i \vec{x}_i = \vec{0} \text{ iff all } a_i \text{ vanish.}$$

(ii) If E' is finite-dimensional, all linear maps on E' are uniformly continuous. (See also Problems 3 and 4 of §6.)

13. Prove that if $f\colon E^1 \to E$ is continuous and $(\forall\, x, y \in E^1)$

$$f(x+y) = f(x) + f(y),$$

then f is linear; so, by Corollary 2, $f(x) = vx$ where $v = f(1)$.

[Hint: Show that $f(ax) = af(x)$; first for $a = 1, 2, \ldots$ (note: $nx = x + x + \cdots + x$, n terms); then for *rational* $a = m/n$; then for $a = 0$ and $a = -1$. *Any* $a \in E^1$ is a limit of rationals; so use continuity and Theorem 1 in Chapter 4, §2.]

§3. Differentiable Functions

As we know, a function $f\colon E^1 \to E$ (*on* E^1) is differentiable at $p \in E^1$ iff, with

$\Delta f = f(x) - f(p)$ and $\Delta x = x - p$,

$$f'(p) = \lim_{x \to p} \frac{\Delta f}{\Delta x} \text{ exists} \quad (\textit{finite}).$$

Setting $\Delta x = x - p = t$, $\Delta f = f(p+t) - f(p)$, and $f'(p) = v$, we may write this equation as

$$\lim_{t \to 0} \left| \frac{\Delta f}{t} - v \right| = 0,$$

or

$$\lim_{t \to 0} \frac{1}{|t|} |f(p+t) - f(p) - vt| = 0. \tag{1}$$

Now define a map $\phi \colon E^1 \to E$ by $\phi(t) = tv$, $v = f'(p) \in E$.

Then ϕ is linear and continuous, i.e., $\phi \in L(E^1, E)$; so by Corollary 2 in §2, we may express (1) as follows: *there is a map* $\phi \in L(E^1, E)$ *such that*

$$\lim_{t \to 0} \frac{1}{|t|} |\Delta f - \phi(t)| = 0.$$

We adopt this as a definition in the general case, $f \colon E' \to E$, as well.

Definition 1.

A function $f \colon E' \to E$ (where E' and E are normed spaces over *the same* scalar field) is said to be *differentiable* at a point $\vec{p} \in E'$ iff there is a map

$$\phi \in L(E', E)$$

such that

$$\lim_{\vec{t} \to \vec{0}} \frac{1}{|\vec{t}|} |\Delta f - \phi(\vec{t})| = 0;$$

that is,

$$\lim_{\vec{t} \to \vec{0}} \frac{1}{|\vec{t}|} [f(\vec{p} + \vec{t}) - f(\vec{p}) - \phi(\vec{t})] = 0. \tag{2}$$

As we show below, ϕ is *unique* (for a fixed $\vec{p}$), if it exists.

We call ϕ the *differential* of f at $\vec{p}$, briefly denoted df. As it depends on $\vec{p}$, we also write $df(\vec{p}; \vec{t})$ for $df(\vec{t})$ and $df(\vec{p}, \cdot\,)$ for df.

Some authors write $f'(\vec{p})$ for $df(\vec{p}, \cdot\,)$ and call it the *derivative* at $\vec{p}$, but we shall not do this (see Preface). Following M. Spivak, however, we shall use "$[f'(\vec{p})]$" for its *matrix*, as follows.

Definition 2.

If $E' = E^n\,(C^n)$ and $E = E^m\,(C^m)$, and $f \colon E' \to E$ is differentiable at $\vec{p}$, we set

$$[f'(\vec{p})] = [df(\vec{p}, \cdot\,)]$$

and call it the *Jacobian matrix* of f at $\vec{p}$.

Note 1. In Chapter 5, §6, we did not define df as a *mapping*. However, if $E' = E^1$, the *function value*

$$df(p;t) = vt = f'(p)\Delta x$$

is as in Chapter 5, §6.

Also, $[f'(p)]$ is a 1×1 matrix with single term $f'(p)$. (Why?) This motivated Definition 2.

Theorem 1 (uniqueness of df). *If $f\colon E' \to E$ is differentiable at $\vec{p}$, then the map ϕ described in Definition* 1 *is unique* (*dependent on f and $\vec{p}$ only*).

Proof. Suppose there is another linear map $g\colon E' \to E$ such that

$$\text{(3)} \qquad \lim_{\vec{t} \to \vec{0}} \frac{1}{|\vec{t}|}[f(\vec{p}+\vec{t}) - f(\vec{p}) - g(\vec{t})] = \lim_{\vec{t} \to \vec{0}} \frac{1}{|\vec{t}|}[\Delta f - g(\vec{t})] = 0.$$

Let $h = \phi - g$. By Corollary 1 in §2, h is linear.

Also, by the triangle law,

$$|h(\vec{t})| = |\phi(\vec{t}) - g(\vec{t})| \le |\Delta f - \phi(\vec{t})| + |\Delta f - g(\vec{t})|.$$

Hence, dividing by $|\vec{t}|$,

$$\left| h\left(\frac{\vec{t}}{|\vec{t}|}\right) \right| = \frac{1}{|\vec{t}|}\,|h(\vec{t})| \le \frac{1}{|\vec{t}|}\,|\Delta f - \phi(\vec{t})| + \frac{1}{|\vec{t}|}\,|\Delta f - g(\vec{t})|.$$

By (3) and (2), the right side expressions tend to 0 as $\vec{t} \to \vec{0}$. Thus

$$\lim_{\vec{t} \to \vec{0}} h\left(\frac{\vec{t}}{|\vec{t}|}\right) = 0.$$

This remains valid also if $\vec{t} \to \vec{0}$ over any *line* through $\vec{0}$, so that $\vec{t}/|\vec{t}|$ remains *constant*, say $\vec{t}/|\vec{t}| = \vec{u}$, where $\vec{u}$ is an arbitrary (but fixed) unit vector.

Then

$$h\left(\frac{\vec{t}}{|\vec{t}|}\right) = h(\vec{u})$$

is constant; so it can tend to 0 only if it *equals* 0, so $h(\vec{u}) = 0$ for any unit vector $\vec{u}$.

Since *any* $\vec{x} \in E'$ can be written as $\vec{x} = |\vec{x}|\,\vec{u}$, linearity yields

$$h(\vec{x}) = |\vec{x}|\,h(\vec{u}) = 0.$$

Thus $h = \phi - g = 0$ on E', and so $\phi = g$ after all, proving the uniqueness of ϕ. $\square$

Theorem 2. *If f is differentiable at $\vec{p}$, then*

(i) *f is continuous at $\vec{p}$;*

(ii) *for any $\vec{u} \neq \vec{0}$, f has the $\vec{u}$-directed derivative*

$$D_{\vec{u}} f(\vec{p}) = df(\vec{p}; \vec{u}).$$

Proof. By assumption, formula (2) holds for $\phi = df(\vec{p}, \cdot\,)$.

Thus, given $\varepsilon > 0$, there is $\delta > 0$ such that, setting $\Delta f = f(\vec{p} + \vec{t}) - f(\vec{p})$, we have

$$\text{(4)} \qquad \frac{1}{|\vec{t}|}|\Delta f - \phi(\vec{t})| < \varepsilon \text{ whenever } 0 < |\vec{t}| < \delta;$$

or, by the triangle law,

$$\text{(5)} \qquad |\Delta f| \leq |\Delta f - \phi(\vec{t})| + |\phi(\vec{t})| \leq \varepsilon|\vec{t}| + |\phi(\vec{t})|, \quad 0 < |\vec{t}| < \delta.$$

Now, by Definition 1, ϕ is linear and continuous; so

$$\lim_{\vec{t} \to \vec{0}} |\phi(\vec{t})| = |\phi(\vec{0})| = 0.$$

Thus, making $\vec{t} \to \vec{0}$ in (5), with ε fixed, we get

$$\lim_{\vec{t} \to \vec{0}} |\Delta f| = 0.$$

As $\vec{t}$ is just another notation for $\Delta \vec{x} = \vec{x} - \vec{p}$, this proves assertion (i).

Next, fix any $\vec{u} \neq \vec{0}$ in E', and substitute $t\vec{u}$ for $\vec{t}$ in (4).

In other words, t is a *real* variable, $0 < t < \delta/|\vec{u}|$, so that $\vec{t} = t\vec{u}$ satisfies $0 < |\vec{t}| < \delta$.

Multiplying by $|\vec{u}|$, we use the linearity of ϕ to get

$$\varepsilon|\vec{u}| > \left|\frac{\Delta f}{t} - \frac{\phi(t\vec{u})}{t}\right| = \left|\frac{\Delta f}{t} - \phi(\vec{u})\right| = \left|\frac{f(\vec{p} + t\vec{u}) - f(\vec{p})}{t} - \phi(\vec{u})\right|.$$

As ε is arbitrary, we have

$$\phi(\vec{u}) = \lim_{t \to 0} \frac{1}{t}[f(\vec{p} + t\vec{u}) - f(\vec{p})].$$

But this is simply $D_{\vec{u}} f(\vec{p})$, by Definition 1 in §1.

Thus $D_{\vec{u}} f(\vec{p}) = \phi(\vec{u}) = df(\vec{p}; \vec{u})$, proving (ii). □

Note 2. If $E' = E^n$ (C^n), Theorem 2(ii) shows that *if f is differentiable at $\vec{p}$, it has the n partials*

$$D_k f(\vec{p}) = df(\vec{p}; \vec{e}_k), \quad k = 1, \ldots, n.$$

But the converse *fails*: the existence of the $D_k f(\vec{p})$ does not even imply continuity, let alone differentiability (see §1). Moreover, we have the following result.

Corollary 1. *If $E' = E^n\,(C^n)$ and if $f\colon E' \to E$ is differentiable at $\vec{p}$, then*

$$df(\vec{p};\vec{t}) = \sum_{k=1}^{n} t_k D_k f(\vec{p}) = \sum_{k=1}^{n} t_k \frac{\partial}{\partial x_k} f(\vec{p}), \tag{6}$$

where $\vec{t} = (t_1, \ldots, t_n)$.

Proof. By definition, $\phi = df(\vec{p}, \cdot\,)$ is a linear map for a fixed $\vec{p}$.

If $E' = E^n$ or C^n, we may use formula (3) of §2, replacing f and $\vec{x}$ by ϕ and $\vec{t}$, and get

$$\phi(\vec{t}) = df(\vec{p};\vec{t}) = \sum_{k=1}^{n} t_k df(\vec{p};\vec{e}_k) = \sum_{k=1}^{n} t_k D_k f(\vec{p})$$

by Note 2. □

Note 3. In classical notation, one writes Δx_k or dx_k for t_k in (6). Thus, omitting $\vec{p}$ and $\vec{t}$, formula (6) is often written as

$$df = \frac{\partial f}{\partial x_1}\,dx_1 + \frac{\partial f}{\partial x_2}\,dx_2 + \cdots + \frac{\partial f}{\partial x_n}\,dx_n. \tag{6$'$}$$

In particular, if $n = 3$, we write x, y, z for x_1, x_2, x_3. This yields

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz \tag{6$''$}$$

(a familiar calculus formula).

Note 4. If the range space E in Corollary 1 is $E^1\,(C)$, then the $D_k f(\vec{p})$ form an n-tuple of *scalars*, i.e., a *vector in* $E^n\,(C^n)$.

In case $f\colon E^n \to E^1$, we denote it by

$$\nabla f(\vec{p}) = (D_1 f(\vec{p}), \ldots, D_n f(\vec{p})) = \sum_{k=1}^{n} \vec{e}_k D_k f(\vec{p}).$$

In case $f\colon C^n \to C$, we replace the $D_k f(\vec{p})$ by their *conjugates* $\overline{D_k f(\vec{p})}$ and set

$$\nabla f(\vec{p}) = \sum_{k=1}^{n} \vec{e}_k \overline{D_k f(\vec{p})}.$$

The vector $\nabla f(\vec{p})$ is called the *gradient* of f ("grad f") at $\vec{p}$.

From (6) we obtain

$$df(\vec{p};\vec{t}) = \sum_{k=1}^{n} t_k D_k f(\vec{p}) = \vec{t} \cdot \nabla f(\vec{p}) \tag{7}$$

(dot product of $\vec{t}$ by $\nabla f(\vec{p})$), provided $f\colon E^n \to E^1$ (or $f\colon C^n \to C$) is differentiable at $\vec{p}$.

This leads us to the following result.

Corollary 2. *A function $f\colon E^n \to E^1$ (or $f\colon C^n \to C$) is differentiable at $\vec{p}$ iff*

$$\lim_{\vec{t}\to\vec{0}} \frac{1}{|\vec{t}|}|f(\vec{p}+\vec{t}) - f(\vec{p}) - \vec{t}\cdot\vec{v}| = 0 \tag{8}$$

for some $\vec{v} \in E^n$ (C^n).

In this case, necessarily $\vec{v} = \nabla f(\vec{p})$ and $\vec{t}\cdot\vec{v} = df(\vec{p};\vec{t})$, $\vec{t} \in E^n$ (C^n).

Proof. If f is differentiable at $\vec{p}$, we may set $\phi = df(\vec{p}, \cdot\,)$ and $\vec{v} = \nabla f(\vec{p})$.

Then by (7),

$$\phi(\vec{t}) = df(\vec{p};\vec{t}) = \vec{t}\cdot\vec{v};$$

so by Definition 1, (8) results.

Conversely, if some $\vec{v}$ satisfies (8), set $\phi(\vec{t}) = \vec{t}\cdot\vec{v}$. Then (8) implies (2), and ϕ is linear and continuous.

Thus by definition, f is differentiable at $\vec{p}$; so (7) holds.

Also, ϕ is a linear functional on E^n (C^n). By Theorem 2(ii) in §2, the $\vec{v}$ in $\phi(\vec{t}) = \vec{t}\cdot\vec{v}$ is unique, as is ϕ.

Thus by (7), $\vec{v} = \nabla f(\vec{p})$ necessarily. $\square$

Corollary 3 (law of the mean)**.** *If $f\colon E^n \to E^1$* (real) *is relatively continuous on a closed segment $L[\vec{p},\vec{q}]$, $\vec{p} \neq \vec{q}$, and differentiable on $L(\vec{p},\vec{q})$, then*

$$f(\vec{q}) - f(\vec{p}) = (\vec{q}-\vec{p})\cdot\nabla f(\vec{x}_0) \tag{9}$$

for some $\vec{x}_0 \in L(\vec{p},\vec{q})$.

Proof. Let

$$r = |\vec{q}-\vec{p}|,\ \ \vec{v} = \frac{1}{r}(\vec{q}-\vec{p}),\ \text{ and } r\vec{v} = (\vec{q}-\vec{p}).$$

By (7) and Theorem 2(ii),

$$D_{\vec{v}}f(\vec{x}) = df(\vec{x};\vec{v}) = \vec{v}\cdot\nabla f(\vec{x})$$

for $\vec{x} \in L(\vec{p},\vec{q})$. Thus by formula (3′) of Corollary 2 in §1,

$$f(\vec{q}) - f(\vec{p}) = rD_{\vec{v}}f(\vec{x}_0) = r\vec{v}\cdot\nabla f(\vec{x}_0) = (\vec{q}-\vec{p})\cdot\nabla f(\vec{x}_0)$$

for some $\vec{x}_0 \in L(\vec{p},\vec{q})$. $\square$

As we know, the mere existence of partials does not imply differentiability. But the existence of *continuous* partials does. Indeed, we have the following theorem.

Theorem 3. *Let $E' = E^n\,(C^n)$.*

If $f\colon E' \to E$ has the partial derivatives $D_k f$ $(k = 1, \ldots, n)$ on all of an open set $A \subseteq E'$, and if the $D_k f$ are continuous at some $\vec{p} \in A$, then f is differentiable at $\vec{p}$.

Proof. With $\vec{p}$ as above, let

$$\phi(\vec{t}) = \sum_{k=1}^{n} t_k D_k f(\vec{p}) \text{ with } \vec{t} = \sum_{k=1}^{n} t_k \vec{e}_k \in E'.$$

Then ϕ is continuous (a polynomial!) and linear (Corollary 2 in §2).

Thus by Definition 1, it remains to show that

$$\lim_{\vec{t} \to \vec{0}} \frac{1}{|\vec{t}|} |\Delta f - \phi(\vec{t})| = 0;$$

that is,

$$\lim_{\vec{t} \in \vec{0}} \frac{1}{|\vec{t}|} \left| f(\vec{p} + \vec{t}) - f(\vec{p}) - \sum_{k=1}^{n} t_k D_k f(\vec{p}) \right| = 0. \tag{10}$$

To do this, fix $\varepsilon > 0$. As A is open and the $D_k f$ are continuous at $\vec{p} \in A$, there is a $\delta > 0$ such that $G_{\vec{p}}(\delta) \subseteq A$ and *simultaneously* (explain this!)

$$(\forall \vec{x} \in G_{\vec{p}}(\delta)) \quad |D_k f(\vec{x}) - D_k f(\vec{p})| < \frac{\varepsilon}{n}, \; k = 1, \ldots, n.$$

Hence for any set $I \subseteq G_{\vec{p}}(\delta)$

$$\sup_{\vec{x} \in I} |D_k f(\vec{x}) - D_k f(\vec{p})| \le \frac{\varepsilon}{n}. \quad \text{(Why?)} \tag{11}$$

Now fix any $\vec{t} \in E'$, $0 < |\vec{t}| < \delta$, and let $\vec{p}_0 = \vec{p}$,

$$\vec{p}_k = \vec{p} + \sum_{i=1}^{k} t_i e_i, \quad k = 1, \ldots, n.$$

Then

$$\vec{p}_n = \vec{p} + \sum_{i=1}^{n} t_i \vec{e}_i = \vec{p} + \vec{t},$$

$|\vec{p}_k - \vec{p}_{k-1}| = |t_k|$, and *all $\vec{p}_k$ lie in $G_{\vec{p}}(\delta)$*, for

$$|\vec{p}_k - \vec{p}| = \left| \sum_{i=1}^{k} t_i e_i \right| = \sqrt{\sum_{i=1}^{k} |t_i|^2} \le \sqrt{\sum_{i=1}^{n} |t_i|^2} = |\vec{t}| < \delta,$$

as required.

As $G_p(\delta)$ is *convex* (Chapter 4, §9), the segments $I_k = L[\vec{p}_{k-1}, \vec{p}_k]$ all lie in $G_{\vec{p}}(\delta) \subseteq A$; and by assumption, f has all partials there.

Hence by Theorem 1 in §1, f is relatively continuous on all I_k.

All this also applies to the functions g_k, defined by

$$(\forall \vec{x} \in E') \quad g_k(\vec{x}) = f(\vec{x}) - x_k D_k f(\vec{p}), \quad k = 1, \ldots, n. \tag{12}$$

(Why?) Here

$$D_k g_k(\vec{x}) = D_k f(\vec{x}) - D_k f(\vec{p}).$$

(Why?)

Thus by Corollary 2 in §1, and (11) above,

$$\begin{aligned} |g_k(\vec{p}_k) - g_k(\vec{p}_{k-1})| &\le |\vec{p}_k - \vec{p}_{k-1}| \sup_{x \in I_k} |D_k f(\vec{x}) - D_k f(\vec{p})| \\ &\le \frac{\varepsilon}{n}|t_k| \le \frac{\varepsilon}{n}|\vec{t}|, \end{aligned}$$

since

$$|\vec{p}_k - \vec{p}_{k-1}| = |t_k \vec{e}_k| \le |\vec{t}|,$$

by construction.

Combine with (12), recalling that the kth coordinates x_k, for $\vec{p}_k$ and $\vec{p}_{k-1}$, *differ by* t_k; so we obtain

$$\begin{aligned} |g_k(\vec{p}_k) - g_k(\vec{p}_{k-1})| &= |f(\vec{p}_k) - f(\vec{p}_{k-1}) - t_k D_k f(\vec{p})| \\ &\le \frac{\varepsilon}{n}|\vec{t}|. \end{aligned} \tag{13}$$

Also,

$$\begin{aligned} \sum_{k=1}^{n} [f(\vec{p}_k) - f(\vec{p}_{k-1})] &= f(\vec{p}_n) - f(\vec{p}_0) \\ &= f(\vec{p} + \vec{t}) - f(\vec{p}) = \Delta f \text{ (see above).} \end{aligned}$$

Thus

$$\begin{aligned} \left| \Delta f - \sum_{k=1}^{n} t_k D_k f(\vec{p}) \right| &= \left| \sum_{k=1}^{n} [f(\vec{p}_k) - f(\vec{p}_{k-1}) - t_k D_k f(\vec{p})] \right| \\ &\le n \cdot \frac{\varepsilon}{n}|\vec{t}| = \varepsilon|\vec{t}|. \end{aligned}$$

As ε is arbitrary, (10) follows, and all is proved. □

Theorem 4. *If* $f\colon E^n \to E^m$ *(or* $f\colon C^n \to C^m$*) is differentiable at* $\vec{p}$*, with* $f = (f_1, \ldots, f_m)$*, then* $[f'(\vec{p})]$ *is an* $m \times n$ *matrix,*

$$[f'(\vec{p})] = [D_k f_i(\vec{p})], \quad i = 1, \ldots, m, \ k = 1, \ldots, n. \tag{14}$$

Proof. By definition, $[f'(\vec{p})]$ is the matrix of the linear map $\phi = df(\vec{p}, \cdot)$, $\phi = (\phi_1, \dots, \phi_m)$. Here

$$\phi(\vec{t}) = \sum_{k=1}^{n} t_k D_k f(\vec{p})$$

by Corollary 1.

As $f = (f_1, \dots, f_m)$, we can compute $D_k f(\vec{p})$ *componentwise* by Theorem 5 of Chapter 5, §1, and Note 2 in §1 to get

$$\begin{aligned} D_k f(\vec{p}) &= (D_k f_1(\vec{p}), \dots, D_k f_m(\vec{p})) \\ &= \sum_{i=1}^{m} e_i' D_k f_i(\vec{p}), \quad k = 1, 2, \dots, n, \end{aligned}$$

where the e_i' are the basic vectors in $E^m\,(C^m)$. (Recall that the $\vec{e}_k$ are the basic vectors in $E^n\,(C^n)$.)

Thus

$$\phi(\vec{t}) = \sum_{i=1}^{m} e_i' \phi_i(\vec{t}).$$

Also,

$$\phi(\vec{t}) = \sum_{k=1}^{n} t_k \sum_{i=1}^{m} e_i' D_k f_i(\vec{p}) = \sum_{i=1}^{m} e_i' \sum_{k=1}^{n} t_k D_k f_i(\vec{p}).$$

The uniqueness of the decomposition (Theorem 2 in Chapter 3, §§1–3) now yields

$$\phi_i(\vec{t}) = \sum_{k=1}^{n} t_k D_k f_i(\vec{p}), \quad i = 1, \dots, m, \quad \vec{t} \in E^n\,(C^n).$$

If here $\vec{t} = \vec{e}_k$, then $t_k = 1$, while $t_j = 0$ for $j \neq k$. Thus we obtain

$$\phi_i(\vec{e}_k) = D_k f_i(\vec{p}), \quad i = 1, \dots, m, \ k = 1, \dots, n.$$

Hence

$$\phi(\vec{e}_k) = (v_{1k}, v_{2k}, \dots, v_{mk}),$$

where

$$v_{ik} = \phi_i(\vec{e}_k) = D_k f_i(\vec{p}).$$

But by Note 3 of §2, $v_{1k}, \dots, v_{mk}$ (written vertically) is the kth column of the $m \times n$ matrix $[\phi] = [f'(\vec{p})]$. Thus formula (14) results indeed. $\square$

In conclusion, let us stress again that while $D_{\vec{u}} f(\vec{p})$ is a *constant*, for a fixed $\vec{p}$, $df(\vec{p}, \cdot)$ is a *mapping*

$$\phi \in L(E', E),$$

especially "tailored" for $\vec{p}$.

The reader should carefully study at least the "arrowed" problems below.

Problems on Differentiable Functions

1. Complete the missing details in the proofs of this section.

2. Verify Note 1. Describe $[f'(\vec{p})]$ for $f\colon E^1 \to E^m$, too. Give examples.

⇒**3.** A map $f\colon E' \to E$ is said to satisfy a *Lipschitz condition* (L) of order $\alpha > 0$ at $\vec{p}$ iff

$$(\exists\,\delta > 0)\ (\exists\,K \in E^1)\ (\forall\,\vec{x} \in G_{\neg\vec{p}}(\delta))\quad |f(\vec{x}) - f(\vec{p})| \le K|\vec{x} - \vec{p}|^{\alpha}.$$

Prove the following.

(i) This implies continuity at $\vec{p}$ (but not conversely; see Problem 7 in Chapter 5, §1).

(ii) L of order > 1 implies differentiability at $\vec{p}$, with $df(\vec{p},\cdot\,) = 0$ on E'.

(iii) Differentiability at $\vec{p}$ implies L of order 1 (apply Theorem 1 in §2 to $\phi = df$).

(iv) If f and g are differentiable at $\vec{p}$, then

$$\lim_{\vec{x}\to\vec{p}} \frac{1}{|\Delta\vec{x}|}|\Delta f|\,|\Delta g| = 0.$$

4. For the functions of Problem 5 in §1, find those $\vec{p}$ at which f is differentiable. Find

$$\nabla f(\vec{p}),\ df(\vec{p},\cdot\,),\ \text{and}\ [f'(\vec{p})].$$

[Hint: Use Theorem 3 and Corollary 1.]

⇒**5.** Prove the following statements.

(i) If $f\colon E' \to E$ is constant on an open globe $G \subset E'$, it is differentiable at each $\vec{p} \in G$, and $df(\vec{p},\cdot\,) = 0$ on E'.

(ii) If the latter holds for each $\vec{p} \in G - Q$ (Q countable), then f is constant on G (even on $\overline{G}$) provided f is relatively continuous there.

[Hint: Given $\vec{p}, \vec{q} \in G$, use Theorem 2 in §1 to get $f(\vec{p}) = f(\vec{q})$.]

6. Do Problem 5 in case G is any open *polygon-connected* set in E'. (See Chapter 4, §9.)

⇒**7.** Prove the following.

(i) If $f, g\colon E' \to E$ are differentiable at $\vec{p}$, so is

$$h = af + bg,$$

for any scalars a, b (if f and g are scalar valued, a and b may be *vectors*); moreover,

$$d(af+bg) = a\,df + b\,dg,$$

i.e.,

$$dh(\vec{p};\vec{t}) = a\,df(\vec{p};\vec{t}) + b\,dg(\vec{p};\vec{t}), \quad \vec{t} \in E'.$$

(ii) In case $f, g\colon E^m \to E^1$ or $C^m \to C$, deduce also that

$$\nabla h(\vec{p}) = a\,\nabla f(\vec{p}) + b\,\nabla g(\vec{p}).$$

⇒**8.** Prove that if $f, g\colon E' \to E^1$ (C) are differentiable at $\vec{p}$, then so are

$$h = gf \text{ and } k = \frac{g}{f}.$$

(the latter, if $f(\vec{p}) \neq 0$). Moreover, with $a = f(\vec{p})$ and $b = g(\vec{p})$, show that

(i) $dh = a\,dg + b\,df$ and

(ii) $dk = (a\,dg - b\,df)/a^2$.

If further $E' = E^n$ (C^n), verify that

(iii) $\nabla h(\vec{p}) = a\nabla g(\vec{p}) + b\nabla f(\vec{p})$ and

(iv) $\nabla k(\vec{p}) = (a\nabla g(\vec{p}) - b\nabla f(\vec{p}))/a^2$.

Prove (i) and (ii) for *vector-valued* g, too.

[Hints: (i) Set $\phi = a\,dg + b\,df$, with a and b as above. Verify that

$$\Delta h - \phi(\vec{t}) = g(\vec{p})(\Delta f - df(\vec{t})) + f(\vec{p})(\Delta g - dg(\vec{t})) + (\Delta f)(\Delta g).$$

Use Problem 3(iv) and Definition 1.

(ii) Let $F(\vec{t}) = 1/f(\vec{t})$. Show that $dF = -df/a^2$. Then apply (i) to gF.]

⇒**9.** Let $f\colon E' \to E^m$ (C^m), $f = (f_1, \ldots, f_m)$. Prove that

(i) f is linear iff all its m components f_k are;

(ii) f is differentiable at $\vec{p}$ iff all f_k are, and then $df = (df_1, \ldots, df_m)$. Hence if f is complex, $df = df_{\text{re}} + i \cdot df_{\text{im}}$.

10. Prove the following statements.

(i) If $f \in L(E', E)$ then f is differentiable on E', and $df(\vec{p}, \cdot\,) = f$, $\vec{p} \in E'$.

(ii) Such is any first-degree monomial, hence any sum of such monomials.

11. Any *rational* function is differentiable in its domain.

[Hint: Use Problems 10(ii), 7, and 8. Proceed as in Theorem 3 in Chapter 4, §3.]

12. Do Problem 8(i) in case g is only *continuous* at $\vec{p}$, and $f(\vec{p})=0$. Find dh.

13. Do Problem 8(i) for *dot products* $h=f\cdot g$ of functions $f,g\colon E'\to E^m\,(C^m)$.

14. Prove the following.

(i) If $\phi\in L(E^n,E^1)$ or $\phi\in L(C^n,C)$, then $\|\phi\|=|\vec{v}|$, with $\vec{v}$ as in §2, Theorem 2(ii).

(ii) If $f\colon E^n\to E^1$ ($f\colon C^n\to C^1$) is differentiable at $\vec{p}$, then

$$\|df(\vec{p},\,\cdot\,)\|=|\nabla f(\vec{p})|.$$

Moreover, in case $f\colon E^n\to E^1$,

$$|\nabla f(\vec{p})|\geq D_{\vec{u}}f(\vec{p})\quad\text{if } |\vec{u}|=1$$

and

$$|\nabla f(\vec{p})|=D_{\vec{u}}f(\vec{p})\quad\text{when } \vec{u}=\frac{\nabla f(\vec{p})}{|\nabla f(\vec{p})|};$$

thus

$$|\nabla f(\vec{p})|=\max_{|\vec{u}|=1}D_{\vec{u}}f(\vec{p}).$$

[Hints: Use the *equality* case in Theorem 4(c') of Chapter 3, §§1–3. Use formula (7), Corollary 2, and Theorem 2(ii).]

15. Show that Theorem 3 holds even if

(i) D_1f is discontinuous at $\vec{p}$, and

(ii) f has partials on $A-Q$ only (Q countable, $\vec{p}\notin Q$), provided f is continuous on A in each of the *last* $n-1$ variables.

[Hint: For $k=1$, formula (13) still results by *definition* of D_1f, if a suitable δ has been chosen.]

***16.** Show that Theorem 3 and Problem 15 apply also to any $f\colon E'\to E$ where E' is n-dimensional with basis $\{\vec{u}_1,\ldots,\vec{u}_n\}$ (see Problem 12 in §2) if we write D_kf for $D_{\vec{u}_k}f$.

[Hints: Assume $|\vec{u}_k|=1$, $1\leq k\leq n$ (if not, replace $\vec{u}_k$ by $\vec{u}_k/|\vec{u}_k|$; show that this yields another *basis*). Modify the proof so that the $\vec{p}_k$ are still in $G_{\vec{p}}(\delta)$. Caution: The standard norm of E^n does not apply here.]

17. Let $f_k\colon E^1\to E^1$ be differentiable at p_k $(k=1,\ldots,n)$. For $\vec{x}=(x_1,\ldots,x_n)\in E^n$, set

$$F(\vec{x})=\sum_{k=1}^{n}f_k(x_k)\text{ and }G(\vec{x})=\prod_{k=1}^{n}f_k(x_k).$$

Show that F and G are differentiable at $\vec{p}=(p_1,\ldots,p_n)$. Express $\nabla F(\vec{p})$ and $\nabla G(\vec{p})$ in terms of the $f_k'(p_k)$.

[Hint: In order to use Problems 7 and 8, replace the f_k by suitable functions defined on E^n. For $\nabla G(\vec{p})$, "imitate" Problem 6 in Chapter 5, §1.]

§4. The Chain Rule. The Cauchy Invariant Rule

To generalize the chain rule (Chapter 5, §1), we consider the composite $h = g \circ f$ of two functions, $f\colon E' \to E''$ and $g\colon E'' \to E$, with E', E'', and E as before.

Theorem 1 (chain rule)**.** *If*

$$f\colon E' \to E'' \text{ and } g\colon E'' \to E$$

are differentiable at $\vec{p}$ and $\vec{q} = f(\vec{p})$, respectively, then

$$h = g \circ f$$

is differentiable at $\vec{p}$, and

$$dh(\vec{p}, \cdot\,) = dg(\vec{q}, \cdot\,) \circ df(\vec{p}, \cdot\,). \tag{1}$$

Briefly: "*The differential of the composite is the composite of differentials.*"

Proof. Let $U = df(\vec{p}, \cdot\,)$, $V = dg(\vec{q}, \cdot\,)$, and $\phi = V \circ U$.

As U and V are linear continuous maps, so is ϕ. We must show that $\phi = dh(\vec{p}, \cdot\,)$.

Here it is more convenient to write $\Delta\vec{x}$ or $\vec{x} - \vec{p}$ for the "$\vec{t}$" of Definition 1 in §3. For brevity, we set (with $\vec{q} = f(\vec{p})$)

$$w(\vec{x}) = \Delta h - \phi(\Delta\vec{x}) = h(\vec{x}) - h(\vec{p}) - \phi(\vec{x} - \vec{p}), \quad \vec{x} \in E', \tag{2}$$

$$u(\vec{x}) = \Delta f - U(\Delta\vec{x}) = f(\vec{x}) - f(\vec{p}) - U(\vec{x} - \vec{p}), \quad \vec{x} \in E', \tag{3}$$

$$v(\vec{y}) = \Delta g - V(\Delta\vec{y}) = g(\vec{y}) - g(\vec{q}) - V(\vec{y} - \vec{q}), \quad \vec{y} \in E''. \tag{4}$$

Then what we have to prove (see Definition 1 in §3) reduces to

$$\lim_{\vec{x} \to \vec{p}} \frac{w(\vec{x})}{|\vec{x} - \vec{p}|} = 0, \tag{5}$$

while the assumed existence of $df(\vec{p}, \cdot\,) = U$ and $dg(\vec{q}, \cdot\,) = V$ can be expressed as

$$\lim_{\vec{x} \to \vec{p}} \frac{u(\vec{x})}{|\vec{x} - \vec{p}|} = 0, \tag{5$'$}$$

and

$$\lim_{\vec{y} \to \vec{q}} \frac{v(\vec{y})}{|\vec{y} - \vec{q}|} = 0, \quad \vec{q} = f(\vec{p}). \tag{5$''$}$$

From (2) and (3), recalling that $h = g \circ f$ and $\phi = V \circ U$, we obtain

$$\begin{aligned} w(\vec{x}) &= g(f(\vec{x})) - g(\vec{q}) - V(U(\vec{x} - \vec{p})) \\ &= g(f(\vec{x})) - g(\vec{q}) - V(f(\vec{x}) - f(\vec{p}) - u(\vec{x})). \end{aligned} \tag{6}$$

Using (4), with $\vec{y} = f(\vec{x})$, and the linearity of V, we rewrite (6) as

$$\begin{aligned} w(\vec{x}) &= g(f(\vec{x})) - g(\vec{q}) - V(f(x) - f(p)) - V(u(x)) \\ &= v(f(\vec{x})) + V(u(\vec{x})). \end{aligned}$$

(Verify!) Thus the desired formula (5) will be proved if we show that

$$\lim_{\vec{x} \to \vec{p}} \frac{V(u(\vec{x}))}{|\vec{x} - \vec{p}|} = 0 \tag{6$'$}$$

and

$$\lim_{\vec{x} \to \vec{p}} \frac{v(f(\vec{x}))}{|\vec{x} - \vec{p}|} = 0. \tag{6$''$}$$

Now, as V is linear and continuous, formula (5′) yields (6′). Indeed,

$$\lim_{\vec{x} \to \vec{p}} \frac{V(u(\vec{x}))}{|\vec{x} - \vec{p}|} = \lim_{\vec{x} \to \vec{p}} V\left(\frac{u(\vec{x})}{|\vec{x} - \vec{p}|}\right) = V(0) = 0$$

by Corollary 2 in Chapter 4, §2. (Why?)

Similarly, (5″) implies (6″) by substituting $\vec{y} = f(\vec{x})$, since

$$|f(\vec{x}) - f(\vec{p})| \le K|\vec{x} - \vec{p}|$$

by Problem 3(iii) in §3. (Explain!) Thus all is proved. □

Note 1 (Cauchy invariant rule)**.** Under the same assumptions, we also have

$$dh(\vec{p}; \vec{t}) = dg(\vec{q}; \vec{s}) \tag{7}$$

if $\vec{s} = df(\vec{p}; \vec{t})$, $\vec{t} \in E'$.

For with U and V as above,

$$dh(\vec{p}, \cdot\,) = \phi = V \circ U.$$

Thus if

$$\vec{s} = df(\vec{p}; \vec{t}) = U(\vec{t}),$$

we have

$$dh(\vec{p}; \vec{t}) = \phi(\vec{t}) = V(U(\vec{t})) = V(\vec{s}) = dg(\vec{q}; \vec{s}),$$

proving (7).

Note 2. If

$$E' = E^n\,(C^n),\ E'' = E^m\,(C^m),\ \text{and } E = E^r\,(C^r)$$

then by Theorem 3 of §2 and Definition 2 in §3, we can write (1) in *matrix form*,

$$[h'(\vec{p})] = [g'(\vec{q})]\,[f'(\vec{p})],$$

resembling Theorem 3 in Chapter 5, §1 (with f and g interchanged). Moreover, we have the following theorem.

Theorem 2. *With all as in Theorem* 1, *let*

$$E' = E^n\,(C^n),\ E'' = E^m\,(C^m),$$

and

$$f = (f_1, \ldots, f_m).$$

Then

$$D_k h(\vec{p}) = \sum_{i=1}^{m} D_i g(\vec{q})\, D_k f_i(\vec{p});$$

or, in classical notation,

$$\frac{\partial}{\partial x_k}\, h(\vec{p}) = \sum_{i=1}^{m} \frac{\partial}{\partial y_i}\, g(\vec{q}) \cdot \frac{\partial}{\partial x_k}\, f_i(\vec{p}), \quad k = 1, 2, \ldots, n. \tag{8}$$

Proof. Fix any basic vector $\vec{e}_k$ in E' and set

$$\vec{s} = df(\vec{p}; \vec{e}_k), \quad \vec{s} = (s_1, \ldots, s_m) \in E''.$$

As f is differentiable at $\vec{p}$, so are its components f_i (Problem 9 in §3), and

$$s_i = df_i(\vec{p}; \vec{e}_k) = D_k f_i(\vec{p})$$

by Theorem 2(ii) in §3. Using also Corollary 1 in §3, we get

$$dg(\vec{q}; \vec{s}) = \sum_{i=1}^{m} s_i D_i g(\vec{q}) = \sum_{i=1}^{m} D_k f_i(\vec{p}) D_i g(\vec{q}).$$

But as $\vec{s} = df(\vec{p}; \vec{e}_k)$, formula (7) yields

$$dg(\vec{q}; \vec{s}) = dh(\vec{p}; \vec{e}_k) = D_k h(\vec{p})$$

by Theorem 2(ii) in §3. Thus the result follows. □

Note 3. Theorem 2 is often called the *chain rule for functions of several variables*. It yields Theorem 3 in Chapter 5, §1, if $m = n = 1$.

In classical calculus one often speaks of derivatives and differentials of *variables* $y = f(x_1, \ldots, x_n)$ rather than those of *mappings*. Thus Theorem 2 is stated as follows.

Let $u = g(y_1, \ldots, y_m)$ *be differentiable. If, in turn, each*

$$y_i = f_i(x_1, \ldots, x_n)$$

is differentiable for $i = 1, \ldots, m$, *then* u *is also differentiable as a composite function of the* n *variables* x_k, *and ("simplifying" formula (8)) we have*

$$\frac{\partial u}{\partial x_k} = \sum_{i=1}^{m} \frac{\partial u}{\partial y_i} \frac{\partial y_i}{\partial x_k}, \quad k = 1, 2, \ldots, n. \tag{8$'$}$$

It is understood that the partials

$$\frac{\partial u}{\partial x_k} \text{ and } \frac{\partial y_i}{\partial x_k} \text{ are taken at some } \vec{p} \in E',$$

while the $\partial u / \partial y_i$ are at $\vec{q} = f(\vec{p})$, where $f = (f_1, \ldots, f_m)$. This "variable" notation is convenient in computations, but may cause ambiguities (see the next example).

Example.

Let $u = g(x, y, z)$, where z *depends on* x *and* y:

$$z = f_3(x, y).$$

Set $f_1(x, y) = x$, $f_2(x, y) = y$, $f = (f_1, f_2, f_3)$, and $h = g \circ f$; so

$$h(x, y) = g(x, y, z).$$

By (8$'$),

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}.$$

Here

$$\frac{\partial x}{\partial x} = \frac{\partial f_1}{\partial x} = 1 \text{ and } \frac{\partial y}{\partial x} = 0,$$

for f_2 does not depend on x. Thus we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}. \tag{9}$$

(Question: Is $\big(\partial u / \partial z\big)\big(\partial z / \partial x\big) = 0$?)

The trouble with (9) is that the variable u "poses" as both g and h. On the left, it is h; on the right, it is g.

To avoid this, our method is to differentiate well-defined *mappings*, not "variables." Thus in (9), we have the maps

$$g\colon E^3 \to E \text{ and } f\colon E^2 \to E^3,$$

with f_1, f_2, f_3 as indicated. Then if $h = g \circ f$, Theorem 2 states (9) *unambiguously* as

$$D_1h(\vec{p}) = D_1g(\vec{q}) + D_3g(\vec{q}) \cdot D_1f(\vec{p}),$$

where $\vec{p} \in E^2$ and

$$\vec{q} = f(\vec{p}) = (p_1, p_2, f_3(\vec{p})).$$

(Why?) In classical notation,

$$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\,\frac{\partial f_3}{\partial x}$$

(avoiding the "paradox" of (9)).

Nonetheless, with due caution, one may use the "variable" notation where convenient. The reader should practice both (see the Problems).

Note 4. The Cauchy rule (7), in "variable" notation, turns into

$$du = \sum_{i=1}^{m} \frac{\partial u}{\partial y_i}\,dy_i = \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}\,dx_k, \tag{10}$$

where $dx_k = t_k$ and $dy_i = df_i(\vec{p}; \vec{t})$.

Indeed, by Corollary 1 in §3,

$$dh(\vec{p}; \vec{t}) = \sum_{k=1}^{n} D_kh(\vec{p}) \cdot t_k \text{ and } dg(\vec{q}; \vec{s}) = \sum_{i=1}^{m} D_ig(\vec{q}) \cdot s_i.$$

Now, in (7),

$$\vec{s} = (s_1, \ldots, s_m) = df(\vec{p}; \vec{t});$$

so by Problem 9 in §3,

$$df_i(\vec{p}; \vec{t}) = s_i, \quad i = 1, \ldots, m.$$

Rewriting all in the "variable" notation, we obtain (10).

The "advantage" of (10) is that du has the *same form*, independently of whether u is treated as a function of the x_k or of the y_i (hence the name *"invariant" rule*). However, one must remember the meaning of dx_k and dy_i, which are quite different.

The "invariance" also fails completely for differentials of higher order (§5).

The advantages of the "variable" notation vanish unless one is able to "translate" it into *precise* formulas.

Further Problems on Differentiable Functions

1. For $E = E^r$ (C^r) prove Theorem 2 *directly*.
[Hint: Find

$$D_k h_j(\vec{p}), \quad j = 1, \ldots, r,$$

from Theorem 4 of §3, and Theorem 3 of §2. Verify that

$$D_k h(\vec{p}) = \sum_{j=1}^{r} e_j D_k h_j(\vec{p}) \text{ and } D_i g(\vec{q}) = \sum_{j=1}^{r} e_j D_i g_j(\vec{q}),$$

where the e_j are the basic unit vectors in E^r. Proceed.]

2. Let $g(x, y, z) = u$, $x = f_1(r, \theta)$, $y = f_2(r, \theta)$, $z = f_3(r, \theta)$, and

$$f = (f_1, f_2, f_3) \colon E^2 \to E^3.$$

Assuming differentiability, verify (using "variables") that

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = \frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial \theta} d\theta$$

by computing derivatives from (8′). Then do all in the mapping notation for $H = g \circ f$, $dH(\vec{p}; \vec{t})$.

3. For the specific functions f, g, h, and k of Problems 4 and 5 of §2, set up and solve problems analogous to Problem 2, using

(a) $k \circ f$; (b) $g \circ k$; (c) $f \circ h$; (d) $h \circ g$.

4. For the functions of Problem 5 in §1, find the formulas for $df(\vec{p}; \vec{t})$. At which $\vec{p}$ does $df(\vec{p}, \cdot)$ exist in each given case? Describe it for a chosen $\vec{p}$.

5. From Theorem 2, with $E = E^1$ (C), find

$$\nabla h(\vec{p}) = \sum_{k=1}^{n} D_k g(\vec{q}) \nabla f_k(\vec{p}).$$

6. Use Theorem 1 for a new solution of Problem 7 in §3 with $E = E^1$ (C).
[Hint: Define F on E' and G on E^2 (C^2) by

$$F(\vec{x}) = (f(\vec{x}), g(\vec{x})) \text{ and } G(\vec{y}) = ay_1 + by_2.$$

Then $h = af + bg = G \circ F$. (Why?) Use Problems 9 and 10(ii) of §3. Do all in "variable" notation, too.]

7. Use Theorem 1 for a new proof of the "only if" in Problem 9 in §3.
[Hint: Set $f_i = g \circ f$, where $g(\vec{x}) = x_i$ (the ith "projection map") is a *monomial*. Verify!]

8. Do Problem 8(i) in §3 for the case $E' = E^2$ (C^2), with

$$f(\vec{x}) = x_1 \text{ and } g(\vec{x}) = x_2.$$

(Simplify!) Then do the general case as in Problem 6 above, with

$$G(\vec{y}) = y_1 y_2.$$

9. Use Theorem 2 for a new proof of Theorem 4 in Chapter 5, §1. (Proceed as in Problems 6 and 8, with $E' = E^1$, so that $D_1 h = h'$.) Do it in the "variable" notation, too.

10. Under proper differentiability assumptions, use formula (8′) to express the partials of u if

(i) $u = g(x, y)$, $x = f(r)h(\theta)$, $y = r + h(\theta) + \theta f(r)$;

(ii) $u = g(r, \theta)$, $r = f(x + f(y))$, $\theta = f(xf(y))$;

(iii) $u = g(x^y, y^z, z^{x+y})$.

Then redo all in the "mapping" terminology, too.

11. Let the map $g\colon E^1 \to E^1$ be differentiable on E^1. Find $|\nabla h(\vec{p})|$ if $h = g \circ f$ and

(i) $f(\vec{x}) = \sum_{k=1}^{n} x_k$, $\vec{x} \in E^n$;

(ii) $f = (f_1, f_2)$, $f_1(\vec{x}) = \sum_{k=1}^{n} x_k$, $f_2(\vec{x}) = |\vec{x}|^2$, $\vec{x} \in E^n$.

12. (Euler's theorem.) A map $f\colon E^n \to E^1$ (or $C^n \to C$) is called *homogeneous of degree* m on G iff

$$(\forall t \in E^1\,(C)) \quad f(t\vec{x}) = t^m f(\vec{x})$$

when $\vec{x}$, $t\vec{x} \in G$. Prove the following statements.

(i) If so, and f is differentiable at $\vec{p} \in G$ (*an open globe*), then

$$\vec{p} \cdot \nabla f(\vec{p}) = mf(\vec{p}).$$

*(ii) Conversely, if the latter holds for all $\vec{p} \in G$ and if $\vec{0} \notin G$, then f is homogeneous of degree m on G.

(iii) What if $\vec{0} \in G$?

[Hints: (i) Let $g(t) = f(t\vec{p})$. Find $g'(1)$. (iii) Take $f(x, y) = x^2 y^2$ if $x \le 0$, $f = 0$ if $x > 0$, $G = G_0(1)$.]

13. Try Problem 12 for $f\colon E' \to E$, replacing $\vec{p} \cdot \nabla f(\vec{p})$ by $df(\vec{p}; \vec{p})$.

14. With all as in Theorem 1, prove the following.

(i) If $E' = E^1$ and $\vec{s} = f'(p) \neq \vec{0}$, then $h'(p) = D_{\vec{s}}\, g(\vec{q})$.

(ii) If $\vec{u}$ and $\vec{v}$ are nonzero in E' and $a\vec{u} + b\vec{v} \neq \vec{0}$ for some scalars a, b, then

$$D_{a\vec{u}+b\vec{v}} f(\vec{p}) = aD_{\vec{u}} f(\vec{p}) + bD_{\vec{v}} f(\vec{p}).$$

(iii) If f is differentiable on a globe $G_{\vec{p}}$, and $\vec{u} \neq \vec{0}$ in E', then

$$D_{\vec{u}} f(\vec{p}) = \lim_{\vec{x} \to \vec{u}} D_{\vec{x}}(\vec{p}).$$

[Hints: Use Theorem 2(ii) from §3 and Note 1.]

15. Use Theorem 2 to find the partially derived functions of f, if

(i) $f(x, y, z) = \big(\sin(xy/z)\big)^x$;

(ii) $f(x, y) = \log_x \big|\tan(y/x)\big|$.

(Set $f = 0$ wherever undefined.)

§5. Repeated Differentiation. Taylor's Theorem

In §1 we defined *$\vec{u}$-directed derived functions*, $D_{\vec{u}} f$ for *any* $f \colon E' \to E$ and *any* $\vec{u} \neq \vec{0}$ in E'.

Thus given a sequence $\{\vec{u}_i\} \subseteq E' - \{\vec{0}\}$, we can first form $D_{\vec{u}_1} f$, then $D_{\vec{u}_2}(D_{\vec{u}_1} f)$ (the $\vec{u}_2$-directed derived function of $D_{\vec{u}_1} f$), then the $\vec{u}_3$-directed derived function of $D_{\vec{u}_2}(D_{\vec{u}_1} f)$, and so on. We call all functions so formed the *higher-order directional derived functions of f*.

If *at each step* the *limit* postulated in Definition 1 of §1 exists for all $\vec{p}$ in a set $B \subseteq E'$, we call them the higher-order directional *derivatives* of f (on B).

If all $\vec{u}_i$ are basic unit vectors in E^n (C^n), we say "*partial*" instead of "directional."

We also define $D^1_{\vec{u}} f = D_{\vec{u}} f$ and

$$D^k_{\vec{u}_1 \vec{u}_2 \ldots \vec{u}_k} f = D_{\vec{u}_k}(D^{k-1}_{\vec{u}_1 \vec{u}_2 \ldots \vec{u}_{k-1}} f), \quad k = 2, 3, \ldots, \tag{1}$$

and call $D^k_{\vec{u}_1 \vec{u}_2 \ldots \vec{u}_k} f$ a directional derived function of *order* k. (Some authors denote it by $D^k_{\vec{u}_k \vec{u}_{k-1} \ldots \vec{u}_1} f$.)

If all $\vec{u}_i$ equal $\vec{u}$, we write $D^k_{\vec{u}} f$ instead.

For *partially* derived functions, we simplify this notation, writing $1\,2\,\ldots$ for $\vec{e}_1 \vec{e}_2 \ldots$ and omitting the "k" in D^k (except in *classical* notation):

$$D_{12} f = D^2_{\vec{e}_1 \vec{e}_2} f = \frac{\partial^2 f}{\partial x_1 \, \partial x_2}, \; D_{11} f = D^2_{\vec{e}_1 \vec{e}_1} f = \frac{\partial^2 f}{\partial x_1^2}, \text{ etc.}$$

We also set $D^0_{\vec{u}} f = f$ for *any* vector $\vec{u}$.

Example.

(A) Define $f \colon E^2 \to E^1$ by

$$f(0, 0) = 0, \quad f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}.$$

Then

$$\frac{\partial f}{\partial x} = D_1 f(x, y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2},$$

whence $D_1 f(0, y) = -y$ if $y \neq 0$; and also

$$D_1 f(0, 0) = \lim_{x \to 0} \frac{f(x, 0) - f(0, 0)}{x} = 0. \quad \text{(Verify!)}$$

Thus $D_1 f(0, y) = -y$ *always*, and so $D_{12} f(0, y) = -1$; $D_{12} f(0, 0) = -1$. Similarly,

$$D_2 f(x, y) = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

if $x \neq 0$ and $D_2 f(0, 0) = 0$. Thus $(\forall\, x)$ $D_2 f(x, 0) = x$ and so

$$D_{21} f(x, 0) = 1 \text{ and } D_{21} f(0, 0) = 1 \neq D_{12} f(0, 0) = -1.$$

The previous example shows that *we may well have* $D_{12} f \neq D_{21} f$, or more generally, $D^2_{\vec{u}\vec{v}} f \neq D^2_{\vec{v}\vec{u}} f$. However, we obtain the following theorem.

Theorem 1. *Given nonzero vectors $\vec{u}$ and $\vec{v}$ in E', suppose $f\colon E' \to E$ has the derivatives*

$$D_{\vec{u}} f,\ D_{\vec{v}} f, \text{ and } D^2_{\vec{u}\vec{v}} f$$

on an open set $A \subseteq E'$.

If $D^2_{\vec{u}\vec{v}} f$ is continuous at some $\vec{p} \in A$, then the derivative $D^2_{\vec{v}\vec{u}} f(\vec{p})$ also exists and equals $D^2_{\vec{u}\vec{v}} f(\vec{p})$.

Proof. By Corollary 1 in §1, all reduces to the case $|\vec{u}| = 1 = |\vec{v}|$. (Why?)

Given $\varepsilon > 0$, fix $\delta > 0$ so small that $G = G_{\vec{p}}(\delta) \subseteq A$ and simultaneously

$$\sup_{\vec{x} \in G} |D^2_{\vec{u}\vec{v}} f(\vec{x}) - D^2_{\vec{u}\vec{v}} f(\vec{p})| \leq \varepsilon \tag{2}$$

(by the continuity of $D^2_{\vec{u}\vec{v}} f$ at $\vec{p}$).

Now $(\forall\, s, t \in E^1)$ define $H_t\colon E^1 \to E$ by

$$H_t(s) = D_{\vec{u}} f(\vec{p} + t\vec{u} + s\vec{v}).$$

Let

$$I = \left(-\frac{\delta}{2}, \frac{\delta}{2}\right).$$

If $s, t \in I$, the point $\vec{x} = \vec{p} + t\vec{u} + s\vec{v}$ is in $G_{\vec{p}}(\delta) \subseteq A$, since

$$|\vec{x} - \vec{p}| = |t\vec{u} + s\vec{v}| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Thus by assumption, the derivative $D^2_{\vec{u}\vec{v}}f(\vec{p})$ exists. Also,

$$\begin{aligned} H_t'(s) &= \lim_{\Delta s \to 0} \frac{1}{\Delta s}[H_t(s + \Delta s) - H_t(s)] \\ &= \lim_{\Delta s \to 0} \frac{1}{\Delta s}[D_{\vec{u}}f(\vec{x} + \Delta s \cdot \vec{v}) - D_{\vec{u}}f(\vec{x})]. \end{aligned}$$

But the last limit is $D^2_{\vec{u}\vec{v}}f(\vec{x})$, *by definition*. Thus, setting

$$h_t(s) = H_t(s) - sD^2_{\vec{u}\vec{v}}f(\vec{p}),$$

we get

$$\begin{aligned} h_t'(s) &= H_t'(s) - D^2_{\vec{u}\vec{v}}f(\vec{p}) \\ &= D^2_{\vec{u}\vec{v}}f(\vec{x}) - D^2_{\vec{u}\vec{v}}f(\vec{p}). \end{aligned}$$

We see that h_t is differentiable on I, and by (2),

$$\sup_{s \in I} |h_t'(s)| \leq \sup_{\vec{x} \in G} |D^2_{\vec{u}\vec{v}}f(\vec{x}) - D^2_{\vec{u}\vec{v}}f(\vec{p})| \leq \varepsilon$$

for all $t \in I$. Hence by Corollary 1 of Chapter 5, §4,

$$|h_t(s) - h_t(0) \leq |s| \sup_{\sigma \in I} |h_t'(\sigma)| \leq |s|\varepsilon.$$

But by definition,

$$h_t(s) = D_{\vec{u}}f(\vec{p} + t\vec{u} + s\vec{v}) - sD^2_{\vec{u}\vec{v}}f(\vec{p})$$

and

$$h_t(0) = D_{\vec{u}}f(\vec{p} + t\vec{u}).$$

Thus

$$|D_{\vec{u}}f(\vec{p} + t\vec{u} + s\vec{v}) - D_{\vec{u}}f(\vec{p} + t\vec{u}) - sD^2_{\vec{u}\vec{v}}f(\vec{p})| \leq |s|\varepsilon \tag{3}$$

for all $s, t \in I$.

Next, set

$$G_s(t) = f(\vec{p} + t\vec{u} + s\vec{v}) - f(\vec{p} + t\vec{u})$$

and

$$g_s(t) = G_s(t) - st \cdot D^2_{\vec{u}\vec{v}}f(\vec{p}).$$

As before, one finds that $(\forall\, s \in I)$ g_s is differentiable on I and that

$$g_s'(t) = D_{\vec{u}}f(\vec{p} + t\vec{u} + s\vec{v}) - D_{\vec{u}}f(\vec{p} + t\vec{u}) - sD^2_{\vec{u}\vec{v}}f(\vec{p})$$

for $s, t \in I$. (Verify!)

Hence by (3),

$$\sup_{t \in I} |g_s'(t)| \leq |s|\varepsilon.$$

Again, by Corollary 1 of Chapter 5, §4,

$$|g_s(t) - g_s(0)| \le |st|\varepsilon,$$

or by the definition of g_s (assuming $s, t \in I - \{0\}$ and dividing by st),

$$\left|\frac{1}{st}[f(\vec{p} + t\vec{u} + s\vec{v}) - f(\vec{p} + t\vec{u})] - D^2_{\vec{u}\vec{v}} f(\vec{p}) - \frac{1}{st}[f(\vec{p} + s\vec{v}) - f(\vec{p})]\right| \le \varepsilon.$$

(Verify!) Making $s \to 0$ (with t fixed), we get, by the definition of $D_{\vec{v}} f$,

$$\left|\frac{1}{t} D_{\vec{v}} f(\vec{p} + t\vec{u}) - \frac{1}{t} D_{\vec{v}} f(\vec{p}) - D^2_{\vec{u}\vec{v}} f(\vec{p})\right| \le \varepsilon$$

whenever $0 < |t| < \delta/2$.

As ε is arbitrary, we have

$$D^2_{\vec{u}\vec{v}} f(\vec{p}) = \lim_{t\to 0} \frac{1}{t}[D_v f(\vec{p} + t\vec{u}) - D_{\vec{v}} f(\vec{p})].$$

But by definition, this limit *is* the derivative $D^2_{\vec{v}\vec{u}} f(\vec{p})$. Thus all is proved. □

Note 1. By induction, the theorem extends to derivatives of order > 2. Thus the derivative $D^k_{\vec{u}_1\vec{u}_2\ldots\vec{u}_k} f$ is independent of the order in which the $\vec{u}_i$ follow each other if it exists and is continuous on an open set $A \subseteq E'$, along with appropriate derivatives of order $< k$.

If $E' = E^n$ (C^n), this applies to *partials* as a special case.

For E^n and C^n *only*, we also formulate the following definition.

Definition 1.

Let $E' = E^n$ (C^n). We say that $f\colon E' \to E$ is *m times differentiable* at $\vec{p} \in E'$ iff f and all its partials of order $< m$ are differentiable at $\vec{p}$.

If this holds for all $\vec{p}$ in a set $B \subseteq E'$, we say that *f is m times differentiable on B*.

If, in addition, all partials of order m are continuous at $\vec{p}$ (on B), we say that f is *of class CD^m*, or *continuously differentiable m times* there, and write $f \in CD^m$ at $\vec{p}$ (on B).

Finally, if this holds for *all* natural m, we write $f \in CD^\infty$ at $\vec{p}$ (on B, respectively).

Definition 2.

Given the space $E' = E^n$ (C^n), the function $f\colon E' \to E$, and a point $\vec{p} \in E'$, we define the mappings

$$d^m f(\vec{p}, \cdot\,), \quad m = 1, 2, \ldots,$$

from E' to E by setting for every $\vec{t} = (t_1, \ldots, t_n)$

$$
\begin{aligned}
d^1 f(\vec{p}; \vec{t}) &= \sum_{i=1}^{n} D_i f(\vec{p}) \cdot t_i, \\
d^2 f(\vec{p}; \vec{t}) &= \sum_{j=1}^{n} \sum_{i=1}^{n} D_{ij} f(\vec{p}) \cdot t_i t_j, \\
d^3 f(\vec{p}; \vec{t}) &= \sum_{k=1}^{n} \sum_{j=1}^{n} \sum_{i=1}^{n} D_{ijk} f(\vec{p}) \cdot t_i t_j t_k, \quad \text{and so on.}
\end{aligned}
\tag{4}
$$

We call $d^m f(\vec{p}, \cdot\,)$ *the mth differential* (*or differential of order* m) of f at $\vec{p}$. By our conventions, it is *always* defined on E^n (C^n) as are the partially derived functions involved.

If f is differentiable at $\vec{p}$ (but not otherwise), then $d^1 f(\vec{p}; \vec{t}) = df(\vec{p}; \vec{t})$ by Corollary 1 in §3; $d^1 f(\vec{p}, \cdot\,)$ is linear and continuous (why?) but need not satisfy Definition 1 in §3.

In classical notation, we write dx_i for t_i; e.g.,

$$d^2 f = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i\, \partial x_j}\, dx_i\, dx_j.$$

Note 2. Classical analysis tends to *define* differentials as above in terms of partials. Formula (4) for $d^m f$ is often written *symbolically*:

$$d^m f = \left(\frac{\partial}{\partial x_1}\, dx_1 + \frac{\partial}{\partial x_2}\, dx_2 + \cdots + \frac{\partial}{\partial x_n}\, dx_n \right)^m f, \quad m = 1, 2, \ldots \tag{5}$$

Indeed, raising the bracketed expression to the mth "power" as in algebra (removing brackets, without collecting "similar" terms) and then "multiplying" by f, we obtain sums that agree with (4). (Of course, this is not genuine multiplication but only a convenient memorizing device.)

Example.

(B) Define $f\colon E^2 \to E^1$ by

$$f(x, y) = x \sin y.$$

Take *any* $\vec{p} = (x, y) \in E^2$. Then

$$
\begin{gathered}
D_1 f(x, y) = \sin y \text{ and } D_2 f(x, y) = x \cos y; \\
D_{12} f(x, y) = D_{21} f(x, y) = \cos y, \\
D_{11} f(x, y) = 0, \text{ and } D_{22} f(x, y) = -x \sin y; \\
D_{111} f(x, y) = D_{112} f(x, y) = D_{121} f(x, y) = D_{211} f(x, y) = 0, \\
D_{221} f(x, y) = D_{212} f(x, y) = D_{122} f(x, y) = -\sin y, \text{ and}
\end{gathered}
$$

$$D_{222}f(x,y) = -x\cos y; \text{ etc.}$$

As is easily seen, f has continuous partials of all orders; so $f \in CD^\infty$ on all of E^2. Also,

$$\begin{aligned} df(\vec{p};\vec{t}) &= t_1 D_1 f(\vec{p}) + t_2 D_2 f(\vec{p}) \\ &= t_1 \sin y + t_2 x \cos y. \end{aligned}$$

In classical notation,

$$\begin{aligned} df = d^1 f &= \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy \\ &= \sin y\, dx + x\cos y\, dy; \\ d^2 f &= \frac{\partial^2 f}{\partial x^2}\, dx^2 + 2\, \frac{\partial^2 f}{\partial x\,\partial y}\, dx\, dy + \frac{\partial^2 f}{\partial y^2}\, dy^2 \\ &= 2\, \cos y\, dx\, dy - x \sin y\, dy^2; \\ d^3 f &= -3\sin y\, dx\, dy^2 - x\cos y\, dy^3; \end{aligned}$$

and so on. (Verify!)

We can now extend Taylor's theorem (Theorem 1 in Chapter 5, §6) to the case $E' = E^n\ (C^n)$.

Theorem 2 (Taylor). *Let $\vec{u} = \vec{x} - \vec{p} \neq \vec{0}$ in $E' = E^n\ (C^n)$.*

If $f\colon E' \to E$ is $m+1$ times differentiable on the line segment

$$I = L[\vec{p},\vec{x}] \subset E'$$

then

$$f(\vec{x}) - f(\vec{p}) = \sum_{i=1}^{m} \frac{1}{i!}\, d^i f(\vec{p};\vec{u}) + R_m,$$

with

$$|R_m| \le \frac{K_m}{(m+1)!},\ K_m \in E^1, \tag{6}$$

and

$$0 \le K_m \le \sup_{\vec{s}\in I} |d^{m+1} f(\vec{s};\vec{u})|. \tag{6'}$$

Proof. Define $g\colon E^1 \to E'$ and $h\colon E^1 \to E$ by $g(t) = \vec{p} + t\vec{u}$ and $h = f \circ g$.

As $E' = E^n\ (C^n)$, we may consider the *components* of g,

$$g_k(t) = p_k + t u_k, \quad k \le n.$$

Clearly, g_k is differentiable, $g_k'(t) = u_k$.

By assumption, so is f on $I = L[\vec{p}, \vec{x}]$. Thus, by the chain rule, $h = f \circ g$ is differentiable on the interval $J = [0, 1] \subset E^1$; for, by definition,

$$\vec{p} + t\vec{u} \in L[\vec{p}, \vec{x}] \text{ iff } t \in [0, 1].$$

By Theorem 2 in §4,

$$h'(t) = \sum_{k=1}^{n} D_k f(\vec{p} + t\vec{u}) \cdot u_k = df(\vec{p} + t\vec{u}; \vec{u}), \quad t \in J. \tag{7}$$

(Explain!)

By assumption (and Definition 1), the $D_k f$ are differentiable on I. Hence, by (7), h' is differentiable on J. Reapplying Theorem 2 in §4, we obtain

$$\begin{aligned} h''(t) &= \sum_{j=1}^{n} \sum_{k=1}^{n} D_{kj} f(\vec{p} + t\vec{u}) \cdot u_k u_j \\ &= d^2 f(\vec{p} + t\vec{u}; \vec{u}), \quad t \in J. \end{aligned}$$

By induction, h is $m + 1$ times differentiable on J, and

$$h^{(i)}(t) = d^i f(\vec{p} + t\vec{u}; \vec{u}), \quad t \in J,\ i = 1, 2, \ldots, m + 1. \tag{8}$$

The differentiability of $h^{(i)}$ $(i \leq m)$ implies its continuity on $J = [0, 1]$.

Thus h satisfies Theorem 1 of Chapter 5, §6 (with $x = 1$, $p = 0$, and $Q = \emptyset$); hence

$$\begin{aligned} h(1) - h(0) &= \sum_{i=1}^{m} \frac{h^{(i)}(0)}{i!} + R_m, \\ |R_m| &\leq \frac{K_m}{(m+1)!}, \quad K_m \in E^1, \\ K_m &\leq \sup_{t \in J} |h^{(m+1)}(t)|. \end{aligned} \tag{9}$$

By construction,

$$h(t) = f(g(t)) = f(\vec{p} + t\vec{u});$$

so

$$h(1) = f(\vec{p} + \vec{u}) = f(\vec{x}) \text{ and } h(0) = f(\vec{p}).$$

Thus using (8) also, we see that (9) implies (6), indeed. □

Note 3. Formula (3′) of Chapter 5, §6, combined with (8), also yields

$$\begin{aligned} R_m &= \frac{1}{m!} \int_0^1 h^{(m+1)}(t) \cdot (1 - t)^m \, dt \\ &= \frac{1}{m!} \int_0^1 d^{m+1} f(\vec{p} + t\vec{u}; \vec{u}) \cdot (1 - t)^m \, dt. \end{aligned}$$

Corollary 1 (the Lagrange form of R_m)**.** *If* $E = E^1$ *in Theorem* 2, *then*

$$R_m = \frac{1}{(m+1)!}\, d^{m+1} f(\vec{s}; \vec{u}) \tag{10}$$

for some $\vec{s} \in L(\vec{p}, \vec{x})$.

Proof. Here the function h defined in the proof of Theorem 2 is *real*; so Theorem $1'$ and formula $(3')$ of Chapter 5, §6 apply. This yields (10). Explain! ☐

Corollary 2. *If* $f\colon E^n\,(C^n) \to E$ *is* m *times differentiable at* $\vec{p}$ *and if* $\vec{u} \neq \vec{0}$ $(\vec{p}, \vec{u} \in E^n\,(C^n))$, *then the derivative* $D_{\vec{u}}^m f(\vec{p})$ *exists and equals* $d^m f(\vec{p}; \vec{u})$.

This follows as in the proof of Theorem 2 (with $t = 0$). For by definition,

$$\begin{aligned} D_{\vec{u}} f(\vec{p}) &= \lim_{s \to 0} \frac{1}{s}\, [f(\vec{p} + s\vec{u}) - f(\vec{p})] \\ &= \lim \frac{1}{s}\, [h(s) - h(0)] \\ &= h'(0) = df(\vec{p}; \vec{u}) \end{aligned}$$

by (7). Induction yields

$$D_{\vec{u}}^m f(\vec{p}) = h^{(m)}(0) = d^m(\vec{p}; \vec{u})$$

by (8). (See Problem 3.)

Example.

(C) Continuing Example (B), fix

$$\vec{p} = (1, 0);$$

thus replace (x, y) by $(1, 0)$ there. Instead, write (x, y) for $\vec{x}$ in Theorem 2. Then

$$\vec{u} = \vec{x} - \vec{p} = (x - 1, y);$$

so

$$u_1 = x - 1 = dx \text{ and } u_2 = y = dy,$$

and we obtain

$$\begin{aligned} df(\vec{p}; \vec{u}) &= D_1 f(1, 0) \cdot (x - 1) + D_2 f(1, 0) \cdot y \\ &= (\sin 0) \cdot (x - 1) + (1 \cdot \cos 0) \cdot y \\ &= y; \\ d^2 f(\vec{p}; \vec{u}) &= D_{11} f(1, 0) \cdot (x - 1)^2 + 2\, D_{12} f(1, 0) \cdot (x - 1)y \\ &\quad + D_{22} f(1, 0) \cdot y^2 \\ &= (0) \cdot (x - 1)^2 + 2\,(\cos 0) \cdot (x - 1)y - (1 \cdot \sin 0) \cdot y^2 \\ &= 2(x - 1)y; \end{aligned}$$

and for all $\vec{s} = (s_1, s_2) \in I$,

$$\begin{aligned} d^3 f(\vec{s}; \vec{u}) &= D_{111} f(s_1, s_2) \cdot (x-1)^3 + 3\, D_{112} f(s_1, s_2) \cdot (x-1)^2 y \\ &\qquad + 3\, D_{122} f(s_1, s_2) \cdot (x-1) y^2 + D_{222} f(s_1, s_2) \cdot y^3 \\ &= -3 \sin s_2 \cdot (x-1) y^2 - s_1 \cos s_2 \cdot y^3. \end{aligned} \tag{10′}$$

Hence by (6) and Corollary 1 (with $m = 2$), noting that $f(\vec{p}) = f(1, 0) = 0$, we get

$$\begin{aligned} f(x, y) &= x \cdot \sin y \\ &= y + (x-1) y + R_2, \end{aligned} \tag{11}$$

where for some $\vec{s} \in I$,

$$R_2 = \frac{1}{3!}\, d^3 f(\vec{s}; \vec{u}) = \frac{1}{6} \left[-3 \sin s_2 \cdot (x-1) y^2 - s_1 \cos s_2 \cdot y^3 \right].$$

As $\vec{s} \in L(\vec{p}, \vec{x})$, where $\vec{p} = (1, 0)$ and $\vec{x} = (x, y)$, s_1 is between 1 and x; so

$$|s_1| \le \max(|x|, 1) \le |x| + 1.$$

Finally, since $|\sin s_2| \le 1$ and $|\cos s_2| \le 1$, we obtain

$$|R_2| \le \frac{1}{6} \left[3\, |x-1| + (|x| + 1)\, |y| \right] y^2.$$

This bounds the maximum error that arises if we use (11) to express $x \sin y$ as a second-degree polynomial in $(x-1)$ and y. (See also Problem 4 and Note 4 below.)

Note 4. Formula (6), briefly

$$\Delta f = \sum_{i=1}^{m} \frac{d^i f}{i!} + R_2,$$

generalizes formula (2) in Chapter 5, §6.

As in Chapter 5, §6, we set

$$P_m(\vec{x}) = f(\vec{p}) + \sum_{i=1}^{m} \frac{1}{i!}\, d^i f(\vec{p}; \vec{x} - \vec{p})$$

and call P_m *the mth Taylor polynomial for f* about $\vec{p}$, treating it as a function of n variables x_k, with $\vec{x} = (x_1, \ldots, x_n)$.

When expanded as in Example (C), formula (6) expresses $f(\vec{x})$ in powers of

$$u_k = x_k - p_k, \quad k = 1, \ldots, n,$$

plus the *remainder term* R_m.

If $f \in CD^\infty$ on some $G_{\vec{p}}$ and if $R_m \to 0$ as $m \to \infty$, we can express $f(\vec{x})$ as a convergent power series

$$f(\vec{x}) = \lim_{m\to\infty} P_m(\vec{x}) = f(\vec{p}) + \sum_{i=1}^{\infty} \frac{1}{i!}\, d^i f(\vec{p}; \vec{x} - \vec{p}).$$

We then say that f *admits a Taylor series about* $\vec{p}$, on $G_{\vec{p}}$.

Problems on Repeated Differentiation and Taylor Expansions

1. Complete all details in the proof of Theorem 1. What is the motivation for introducing the auxiliary functions h_t and g_s in this particular way?

2. Is symbolic "multiplication" in Note 2 always commutative? (See Example (A).) Why was it possible to collect "similar" terms

$$\frac{\partial^2 f}{\partial x\, \partial y}\, dx\, dy \text{ and } \frac{\partial^2 f}{\partial y\, \partial x}\, dy\, dx$$

in Example (B)? Using (5), find the *general* formula for $d^3 f$. Expand it!

3. Carry out the induction in Theorem 2 and Corollary 2. (Use a suitable notation for subscripts: $k_1 k_2 \ldots$ instead of $jk \ldots$.)

4. Do Example (C) with $m = 3$ (instead of $m = 2$) and with $\vec{p} = (0, 0)$. Show that $R_m \to 0$, i.e., f admits a Taylor series about $\vec{p}$.

Do it in the following two ways.

(i) Use Theorem 2.

(ii) Expand $\sin y$ as in Problem 6(a) in Chapter 5, §6, and then multiply termwise by x.

Give an estimate for R_3.

5. Use Theorem 2 to expand the following functions in powers of $x - 3$ and $y + 2$ *exactly* (choosing m so that $R_m = 0$).

(i) $f(x, y) = 2xy^2 - 3y^3 + yx^2 - x^3$;

(ii) $f(x, y) = x^4 - x^3y^2 + 2xy - 1$;

(iii) $f(x, y) = x^5 y - axy^5 - x^3$.

6. For the functions of Problem 15 in §4, give their Taylor expansions up to R_2, with

$$\vec{p} = \left(1, \frac{\pi}{4}, 1\right)$$

in case (i) and

$$\vec{p} = \left(e, \frac{\pi}{4} e\right)$$

in (ii). Bound R_2.

7. (Generalized Taylor theorem.) Let $\vec{u} = \vec{x} - \vec{p} \neq \vec{0}$ in E' (E' need not be E^n or C^n); let $I = L[\vec{p}, \vec{x}]$. Prove the following statement:

If $f\colon E' \to E$ and the derived functions $D^i_{\vec{u}} f$ $(i \leq m)$ are relatively continuous on I and have $\vec{u}$-directed derivatives on $I - Q$ (Q countable), then formula (6) and Note 3 hold, with $d^i f(\vec{p}; \vec{u})$ replaced by $D^i_{\vec{u}} f(\vec{p})$.

[Hint: Proceed as in Theorem 2 *without* using the chain rule or any partials or components. Instead of (8), prove that $h^{(i)}(t) = D^i_{\vec{u}} f(\vec{p} + t\vec{u})$ *on* $J - Q'$, $Q' = g^{-1}[Q]$.]

8. (i) Modify Problem 7 by setting

$$\vec{u} = \frac{\vec{x} - \vec{p}}{|\vec{x} - \vec{p}|}.$$

Thus expand $f(\vec{x})$ in powers of $|\vec{x} - \vec{p}|$.

(ii) Deduce Theorem 2 from Problem 7, using Corollary 2.

9. Given $f\colon E^2\,(C^2) \to E$, $f \in CD^m$ on an open set A, and $\vec{s} \in A$, prove that $(\forall\, \vec{u} \in E^2\,(C^2))$

$$d^i f(\vec{s}; \vec{u}) = \sum_{j=0}^{i} \binom{i}{j} u_1^j u_2^{i-j}\, D_{k_1 \ldots k_i} f(\vec{s}), \quad 1 \leq i \leq m,$$

where the $\binom{i}{j}$ are binomial coefficients, and in the jth term,

$$k_1 = k_2 = \cdots = k_j = 2$$

and

$$k_{j+1} = \cdots = k_i = 1.$$

Then restate formula (6) for $n = 2$.

[Hint: Use induction, as in the binomial theorem.]

⇒**10.** Given $\vec{p} \in E' = E^n\,(C^n)$ and $f\colon E' \to E$, prove that $f \in CD^1$ at $\vec{p}$ iff f is differentiable at $\vec{p}$ and

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, \vec{x} \in G_{\vec{p}}(\delta)) \quad \|d^1 f(\vec{p}, \cdot\,) - d^1 f(\vec{x}, \cdot\,)\| < \varepsilon,$$

with norm $\|\ \|$ as in Definition 2 in §2. (*Does* it apply?)

[Hint: If $f \in CD^1$, use Theorem 2 in §3. For the converse, verify that

$$\varepsilon \geq |d^1 f(\vec{p}; \vec{t}) - d^1 f(\vec{x}; \vec{t})| = \left| \sum_{k=1}^{n} [D_k f(\vec{p}) - D_k f(\vec{x})] t_k \right|$$

if $\vec{x} \in G_{\vec{p}}(\delta)$ and $|\vec{t}| \leq 1$. Take $\vec{t} = \vec{e}_k$, to prove continuity of $D_k f$ at $\vec{p}$.]

11. Prove the following.

(i) If $\phi\colon E^n \to E^m$ is linear and $[\phi] = (v_{ik})$, then

$$\|\phi\|^2 \leq \sum_{i,k} |v_{ik}|^2.$$

(ii) If $f\colon E^n \to E^m$ is differentiable at $\vec{p}$, then

$$\|df(\vec{p}, \cdot\,)\|^2 \le \sum_{i,k} |D_k f_i(\vec{p})|^2.$$

(iii) Hence find a new *converse* proof in Problem 10 for $f\colon E^n \to E^m$.

Consider $f\colon C^n \to C^m$, too.

[Hints: (i) By the Cauchy–Schwarz inequality, $|\phi(\vec{x})|^2 \le |\vec{x}|^2 \sum_{i,k} |v_{ik}|^2$. (Why?) (ii) Use part (i) and Theorem 4 in §3.]

12. (i) Find d^2u for the functions of Problem 10 in §4, in the "variable" and "mapping" notations.

(ii) Do it also for

$$u = f(x, y, z) = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$$

and show that $D_{11}f + D_{22}f + D_{33}f = 0$.

(iii) Does the latter hold for $u = \arctan \dfrac{y}{x}$?

13. Let $u = g(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$ (passage to polars).

Using "variables" and then the "mappings" notation, prove that if g is differentiable, then

(i) $\dfrac{\partial u}{\partial r} = \cos\theta\,\dfrac{\partial u}{\partial x} + \sin\theta\,\dfrac{\partial u}{\partial y}$ and

(ii) $|\nabla g(x, y)|^2 = \left(\dfrac{\partial u}{\partial r}\right)^2 + \left(\dfrac{1}{r}\,\dfrac{\partial u}{\partial \theta}\right)^2$.

(iii) Assuming $g \in CD^2$, express $\dfrac{\partial^2 u}{\partial r\,\partial\theta}$, $\dfrac{\partial^2 u}{\partial r^2}$, and $\dfrac{\partial^2 u}{\partial \theta^2}$ as in (i).

14. Let $f, g\colon E^1 \to E^1$ be of class CD^2 on E^1. Verify (in "variable" notation, too) the following statements.

(i) $D_{11}h = a^2 D_{22}h$ if $a \in E^1$ (fixed) and

$$h(x, y) = f(ax + y) + g(y - ax).$$

(ii) $x^2 D_{11}h(x, y) + 2xy\,D_{12}h(x, y) + y^2 D_{22}h(x, y) = 0$ if

$$h(x, y) = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right).$$

(iii) $D_1h \cdot D_{21}h = D_2h \cdot D_{11}h$ if

$$h(x, y) = g(f(x) + y).$$

Find $D_{12}h$, too.

15. Assume $E' = E^n\,(C^n)$ and $E'' = E^m\,(C^m)$. Let $f\colon E' \to E''$ and $g\colon E'' \to E$ be twice differentiable at $\vec{p} \in E'$ and $\vec{q} = f(\vec{p}) \in E''$, respectively, and set $h = g \circ f$.

Show that h is twice differentiable at $\vec{p}$, and

$$d^2h(\vec{p};\vec{t}) = d^2g(\vec{q};\vec{s}) + dg(\vec{q};\vec{v}),$$

where $\vec{t} \in E'$, $\vec{s} = df(\vec{p};\vec{t})$, and $\vec{v} = (v_1, \ldots, v_m) \in E''$ satisfies

$$v_i = d^2f_i(\vec{p};\vec{t}), \quad i = 1, \ldots, m.$$

Thus the second differential is *not* invariant in the sense of Note 4 in §4.
[Hint: Show that

$$D_{kl}h(\vec{p}) = \sum_{j=1}^{m}\sum_{i=1}^{m} D_{ij}g(\vec{q})D_kf_i(\vec{p})D_lf_j(\vec{p}) + \sum_{i=1}^{m} D_ig(\vec{q})D_{kl}f_i(\vec{p}).$$

Proceed.]

16. Continuing Problem 15, prove the invariant rule:

$$d^rh(\vec{p};\vec{t}) = d^rg(\vec{q};\vec{s}),$$

if f is a first-degree polynomial and g is r times differentiable at $\vec{q}$.
[Hint: Here all higher-order partials of f vanish. Use induction.]

§6. Determinants. Jacobians. Bijective Linear Operators

We assume the reader to be familiar with elements of linear algebra. Thus we only briefly recall some definitions and well-known rules.

Definition 1.

Given a linear operator $\phi\colon E^n \to E^n$ (or $\phi\colon C^n \to C^n$), with matrix

$$[\phi] = (v_{ik}), \quad i, k = 1, \ldots, n,$$

we define the *determinant of* $[\phi]$ by

$$\det[\phi] = \det(v_{ik}) = \begin{vmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ v_{21} & v_{22} & \ldots & v_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \ldots & v_{nn} \end{vmatrix} \tag{1}$$

$$= \sum (-1)^{\lambda} v_{1k_1}v_{2k_2}\ldots v_{nk_n},$$

where the sum is over all ordered n-tuples $(k_1, \ldots, k_n)$ of *distinct* integers k_j $(1 \le k_j \le n)$, and

$$\lambda = \begin{cases} 0 & \text{if } \prod_{j<m}(k_m - k_j) > 0 \text{ and} \\ 1 & \text{if } \prod_{j<m}(k_m - k_j) < 0. \end{cases}$$

Recall (Problem 12 in §2) that a set $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in a vector space E is a *basis* iff

(i) B *spans* E, i.e., each $\vec{v} \in E$ has the form

$$\vec{v} = \sum_{i=1}^{n} a_i \vec{v}_i$$

for some scalars a_i, and

(ii) this representation is *unique*.

The latter is true iff the $\vec{v}_i$ are *independent*, i.e.,

$$\sum_{i=1}^{n} a_i \vec{v}_i = \vec{0} \iff a_i = 0, \; i = 1, \ldots, n.$$

If E has a basis of n vectors, we call E *n-dimensional* (e.g., E^n and C^n).

Determinants and bases satisfy the following rules.

(a) *Multiplication rule.* If $\phi, g \colon E^n \to E^n$ (or $C^n \to C^n$) are linear, then

$$\det[g] \cdot \det[\phi] = \det([g]\,[\phi]) = \det[g \circ \phi]$$

(see §2, Theorem 3 and Note 4).

(b) If $\phi(\vec{x}) = \vec{x}$ (*identity map*), then $[\phi] = (v_{ik})$, where

$$v_{ik} = \begin{cases} 0 & \text{if } i \ne k \text{ and} \\ 1 & \text{if } i = k; \end{cases}$$

hence $\det[\phi] = 1$. (Why?) See also the Problems.

(c) An n-dimensional space E is spanned by a set of n vectors *iff* they are independent. If so, each basis consists of exactly n vectors.

Definition 2.

For any function $f \colon E^n \to E^n$ (or $f \colon C^n \to C^n$), we define the *f-induced Jacobian map* $J_f \colon E^n \to E^1$ $(J_f \colon C^n \to C)$ by setting

$$J_f(\vec{x}) = \det(v_{ik}),$$

where $v_{ik} = D_k f_i(\vec{x})$, $\vec{x} \in E^n$ (C^n), and $f = (f_1, \ldots, f_n)$.

The determinant

$$J_f(\vec{p}) = \det(D_k f_i(\vec{p}))$$

is called the *Jacobian* of f at $\vec{p}$.

By our conventions, it is *always* defined, as are the functions $D_k f_i$.

Explicitly, $J_f(\vec{p})$ is the determinant of the right-side matrix in formula (14) in §3. Briefly,

$$J_f = \det(D_k f_i).$$

By Definition 2 and Note 2 in §5,

$$J_f(\vec{p}) = \det[d^1 f(\vec{p}, \cdot\,)].$$

If f is differentiable at $\vec{p}$,

$$J_f(\vec{p}) = \det[f'(\vec{p})].$$

Note 1. More generally, given any functions $v_{ik} \colon E' \to E^1\,(C)$, we can define a map $f \colon E' \to E^1\,(C)$ by

$$f(\vec{x}) = \det(v_{ik}(\vec{x}));$$

briefly $f = \det(v_{ik})$, $i, k = 1, \ldots, n$.

We then call f a *functional determinant.*

If $E' = E^n\,(C^n)$ then f is a function of n variables, since $\vec{x} = (x_1, x_2, \ldots, x_n)$. If all v_{ik} are continuous or differentiable at some $\vec{p} \in E'$, so is f; for by (1), f is a finite sum of functions of the form

$$(-1)^\lambda v_{ik_1} v_{ik_2} \ldots v_{ik_n},$$

and each of these is continuous or differentiable if the v_{ik_i} are (see Problems 7 and 8 in §3).

Note 2. Hence the *Jacobian map* J_f *is continuous or differentiable at* $\vec{p}$ *if all the partially derived functions* $D_k f_i$ $(i, k \leq n)$ *are.*

If, in addition, $J_f(\vec{p}) \neq 0$, then $J_f \neq 0$ *on some globe about* $\vec{p}$. (Apply Problem 7 in Chapter 4, §2, to $|J_f|$.)

In classical notation, one writes

$$\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)} \text{ or } \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)}$$

for $J_f(\vec{x})$. Here $(y_1, \ldots, y_n) = f(x_1, \ldots, x_n)$.

The remarks made in §4 apply to this "variable" notation too. The chain rule easily yields the following corollary.

Corollary 1. *If* $f \colon E^n \to E^n$ *and* $g \colon E^n \to E^n$ *(or* $f, g \colon C^n \to C^n$*) are differentiable at* $\vec{p}$ *and* $\vec{q} = f(\vec{p})$*, respectively, and if*

$$h = g \circ f,$$

then

(i) $$J_h(\vec{p}) = J_g(\vec{q}) \cdot J_f(\vec{p}) = \det(z_{ik}),$$

where

$$z_{ik} = D_k h_i(\vec{p}), \quad i, k = 1, \ldots, n;$$

or, setting

$$\begin{aligned}(u_1, \ldots, u_n) &= g(y_1, \ldots, y_n) \textit{ and}\\ (y_1, \ldots, y_n) &= f(x_1, \ldots, x_n) \text{ (``}\textit{variables}\text{''}),\end{aligned}$$

we have

(ii) $$\frac{\partial(u_1, \ldots, u_n)}{\partial(x_1, \ldots, x_n)} = \frac{\partial(u_1, \ldots, u_n)}{\partial(y_1, \ldots, y_n)} \cdot \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)} = \det(z_{ik}),$$

where

$$z_{ik} = \frac{\partial u_i}{\partial x_k}, \quad i, k = 1, \ldots, n.$$

Proof. By Note 2 in §4,

$$[h'(\vec{p})] = [g'(\vec{q})] \cdot [f'(\vec{p})].$$

Thus by rule (a) above,

$$\det[h'(\vec{p})] = \det[g'(\vec{q})] \cdot \det[f'(\vec{p})],$$

i.e.,

$$J_h(\vec{p}) = J_g(\vec{q}) \cdot J_f(\vec{p}).$$

Also, if $[h'(\vec{p})] = (z_{ik})$, Definition 2 yields $z_{ik} = D_k h_i(\vec{p})$.

This proves (i), hence (ii) also. ☐

In practice, Jacobians mostly occur when a change of variables is made. For instance, in E^2, we may pass from Cartesian coordinates (x, y) to another system (u, v) such that

$$x = f_1(u, v) \text{ and } y = f_2(u, v).$$

We then set $f = (f_1, f_2)$ and obtain $f\colon E^2 \to E^2$,

$$J_f = \det(D_k f_i), \quad k, i = 1, 2.$$

Example (passage to polar coordinates)**.**

Let $x = f_1(r, \theta) = r \cos\theta$ and $y = f_2(r, \theta) = r \sin\theta$.

Then using the "variable" notation, we obtain $J_f(r, \theta)$ as

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

Thus here $J_f(r, \theta) = r$ for all $r, \theta \in E^1$; J_f is independent of θ.

We now concentrate on *one-to-one* (invertible) functions.

Theorem 1. *For a linear map* $\phi\colon E^n \to E^n$ *(or* $\phi\colon C^n \to C^n$*), the following are equivalent:*

(i) *ϕ is one-to-one;*

(ii) *the column vectors $\vec{v}_1, \ldots, \vec{v}_n$ of the matrix $[\phi]$ are independent;*

(iii) *ϕ is onto E^n (C^n);*

(iv) $\det[\phi] \neq 0$.

Proof. Assume (i) and let

$$\sum_{k=1}^{n} c_k \vec{v}_k = \vec{0}.$$

To deduce (ii), we must show that all c_k vanish.

Now, by Note 3 in §2, $\vec{v}_k = \phi(\vec{e}_k)$; so by linearity,

$$\sum_{k=1}^{n} c_k \vec{v}_k = \vec{0}$$

implies

$$\phi\left(\sum_{k=1}^{n} c_k \vec{e}_k\right) = \vec{0}.$$

As ϕ is one-to-one, it can vanish at $\vec{0}$ only. Thus

$$\sum_{k=1}^{n} c_k \vec{e}_k = \vec{0}.$$

Hence by Theorem 2 in Chapter 3, §§1–3, $c_k = 0$, $k = 1, \ldots, n$, and (ii) follows.

Next, assume (ii); so, by rule (c) above, $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis.

Thus *each* $\vec{y} \in E^n$ (C^n) has the form

$$\vec{y} = \sum_{k=1}^{n} a_k \vec{v}_k = \sum_{k=1}^{n} a_k \phi(\vec{e}_k) = \phi\left(\sum_{k=1}^{n} a_k \vec{e}_k\right) = \phi(\vec{x}),$$

where

$$\vec{x} = \sum_{k=1}^{n} a_k \vec{e}_k \ (\textit{uniquely}).$$

Hence (ii) implies *both* (iii) and (i). (Why?)

Now assume (iii). Then *each* $\vec{y} \in E^n\ (C^n)$ has the form $\vec{y} = \phi(\vec{x})$, where

$$\vec{x} = \sum_{k=1}^{n} x_k \vec{e}_k,$$

by Theorem 2 in Chapter 3, §§1–3. Hence again

$$\vec{y} = \sum_{k=1}^{n} x_k \phi(\vec{e}_k) = \sum_{k=1}^{n} x_k \vec{v}_k;$$

so the $\vec{v}_k$ span *all* of $E^n\ (C^n)$. By rule (c) above, this implies (ii), hence (i), too. Thus (i), (ii), and (iii) are equivalent.

Also, by rules (a) and (b), we have

$$\det[\phi] \cdot \det[\phi^{-1}] = \det[\phi \circ \phi^{-1}] = 1$$

if ϕ is one-to-one (for $\phi \circ \phi^{-1}$ is the *identity map*). Hence $\det[\phi] \neq 0$ if (i) holds.

For the converse, suppose ϕ is *not* one-to-one. Then by (ii), the $\vec{v}_k$ are *not* independent. Thus one of them is a linear combination of the others, say,

$$\vec{v}_1 = \sum_{k=2}^{n} a_k \vec{v}_k.$$

But by linear algebra (Problem 13(iii)), $\det[\phi]$ does not change if $\vec{v}_1$ is replaced by

$$\vec{v}_1 - \sum_{k=2}^{n} a_k \vec{v}_k = \vec{0}.$$

Thus $\det[\phi] = 0$ (one column turning to $\vec{0}$). This completes the proof. □

Note 3. Maps that are both onto and one-to-one are called *bijective*. Such is ϕ in Theorem 1. This means that the equation

$$\phi(\vec{x}) = \vec{y}$$

has a unique solution

$$\vec{x} = \phi^{-1}(\vec{y})$$

for each $\vec{y}$. Componentwise, by Theorem 1, *the equations*

$$\sum_{k=1}^{n} x_k v_{ik} = y_i, \quad i = 1, \ldots, n,$$

have a unique solution for the x_k *iff* $\det(v_{ik}) \neq 0$.

Corollary 2. *If* $\phi \in L(E', E)$ *is bijective, with* E' *and* E complete, *then* $\phi^{-1} \in L(E, E')$.

Proof for $E = E^n\,(C^n)$.[1] The notation $\phi \in L(E', E)$ means that $\phi\colon E' \to E$ is linear and continuous.

As ϕ is bijective, $\phi^{-1}\colon E \to E'$ is *linear* (Problem 12).

If $E = E^n\,(C^n)$, it is *continuous*, too (Theorem 2 in §2).

Thus $\phi^{-1} \in L(E, E')$. □

Note. The case $E = E^n\,(C^n)$ suffices for an undergraduate course. (The beginner is advised to omit the "starred" §8.) Corollary 2 and Theorem 2 below, however, are valid in the general case. So is Theorem 1 in §7.

Theorem 2. *Let* E, E' *and* ϕ *be as in Corollary* 2. *Set*

$$\|\phi^{-1}\| = \frac{1}{\varepsilon}.$$

Then any map $\theta \in L(E', E)$ *with* $\|\theta - \phi\| < \varepsilon$ *is one-to-one, and* θ^{-1} *is uniformly continuous.*

Proof. By Corollary 2, $\phi^{-1} \in L(E, E')$, so $\|\phi^{-1}\|$ is *defined* and > 0 (for ϕ^{-1} is not the zero map, being one-to-one).

Thus we may set

$$\varepsilon = \frac{1}{\|\phi^{-1}\|}, \quad \|\phi^{-1}\| = \frac{1}{\varepsilon}.$$

Clearly $\vec{x} = \phi^{-1}(\vec{y})$ if $\vec{y} = \phi(\vec{x})$. Also,

$$|\phi^{-1}(\vec{y})| \leq \frac{1}{\varepsilon}|\vec{y}|$$

by Note 5 in §2. Hence

$$|\vec{y}| \geq \varepsilon|\phi^{-1}(\vec{y})|,$$

i.e.,

$$|\phi(\vec{x})| \geq \varepsilon|\vec{x}| \tag{2}$$

for all $\vec{x} \in E'$ and $\vec{y} \in E$.

Now suppose $\phi \in L(E', E)$ and $\|\theta - \phi\| = \sigma < \varepsilon$.

Obviously, $\theta = \phi - (\phi - \theta)$, and by Note 5 in §2,

$$|(\phi - \theta)(\vec{x})| \leq \|\phi - \theta\|\,|\vec{x}| = \sigma|\vec{x}|.$$

---

[1] See *§8 for the general case.

Thus for every $\vec{x} \in E'$,

$$\begin{aligned} |\theta(\vec{x})| &\geq |\phi(\vec{x})| - |(\phi - \theta)(\vec{x})| \\ &\geq |\phi(\vec{x})| - \sigma|\vec{x}| \\ &\geq (\varepsilon - \sigma)|\vec{x}| \end{aligned} \tag{3}$$

by (2). Therefore, given $\vec{p} \neq \vec{r}$ in E' and setting $\vec{x} = \vec{p} - \vec{r} \neq \vec{0}$, we obtain

$$|\theta(\vec{p}) - \theta(\vec{r})| = |\theta(\vec{p} - \vec{r})| = |\theta(\vec{x})| \geq (\varepsilon - \sigma)|\vec{x}| > 0 \tag{4}$$

(since $\sigma < \varepsilon$).

We see that $\vec{p} \neq \vec{r}$ implies $\theta(\vec{p}) \neq \theta(\vec{r})$; so θ is one-to-one, indeed.

Also, setting $\theta(\vec{x}) = \vec{z}$ and $\vec{x} = \theta^{-1}(\vec{z})$ in (3), we get

$$|\vec{z}| \geq (\varepsilon - \sigma)|\theta^{-1}(\vec{z})|;$$

that is,

$$|\theta^{-1}(\vec{z})| \leq (\varepsilon - \sigma)^{-1}|\vec{z}| \tag{5}$$

for all $\vec{z}$ in the range of θ (domain of θ^{-1}).

Thus θ^{-1} is linearly bounded (by Theorem 1 in §2), hence uniformly continuous, as claimed. $\square$

Corollary 3. *If $E' = E = E^n$ (C^n) in Theorem 2 above, then for given ϕ and $\delta > 0$, there always is $\delta' > 0$ such that*

$$\|\theta - \phi\| < \delta' \textit{ implies } \|\theta^{-1} - \phi^{-1}\| < \delta.$$

In other words, the transformation $\phi \to \phi^{-1}$ is continuous on $L(E)$, $E = E^n$ (C^n).

Proof. First, since $E' = E = E^n$ (C^n), θ is *bijective* by Theorem 1(iii), so $\theta^{-1} \in L(E)$.

As before, set $\|\theta - \phi\| = \sigma < \varepsilon$.

By Note 5 in §2, formula (5) above implies that

$$\|\theta^{-1}\| \leq \frac{1}{\varepsilon - \sigma}.$$

Also,

$$\phi^{-1} \circ (\theta - \phi) \circ \theta^{-1} = \phi^{-1} - \theta^{-1}$$

(see Problem 11).

Hence by Corollary 4 in §2, recalling that $\|\phi^{-1}\| = 1/\varepsilon$, we get

$$\|\theta^{-1} - \phi^{-1}\| \leq \|\phi^{-1}\| \cdot \|\theta - \phi\| \cdot \|\theta^{-1}\| \leq \frac{\sigma}{\varepsilon(\varepsilon - \sigma)} \to 0 \text{ as } \sigma \to 0. \quad \square$$

Problems on Bijective Linear Maps and Jacobians

1. (i) Can a functional determinant $f = \det(v_{ik})$ (see Note 1) be continuous or differentiable even if the functions v_{ik} are *not*?

(ii) Must a Jacobian map J_f be continuous or differentiable if f is?

Give proofs or counterexamples.

⇒**2.** Prove rule (b) on determinants. More generally, show that if $f(\vec{x}) = \vec{x}$ on an open set $A \subseteq E^n$ (C^n), then $J_f = 1$ on A.

3. Let $f\colon E^n \to E^n$ (or $C^n \to C^n$), $f = (f_1, \ldots, f_n)$.

Suppose *each* f_k *depends on* x_k *only*, i.e.,

$$f_k(\vec{x}) = f_k(\vec{y}) \text{ if } x_k = y_k,$$

regardless of the other coordinates x_i, y_i. Prove that $J_f = \prod_{k=1}^{n} D_k f_k$.
[Hint: Show that $D_k f_i = 0$ if $i \neq k$.]

4. In Corollary 1, show that

$$J_h(\vec{p}) = \prod_{k=1}^{n} D_k f_k(\vec{p}) \cdot J_g(\vec{q})$$

if f also has the property specified in Problem 3. Then do all in "variables," with $y_k = y_k(x_k)$ instead of f_k.

5. Let $E' = E^1$ in Note 1. Prove that if all the v_{ik} are differentiable at p, then $f'(p)$ is the sum of n determinants, each arising from $\det(v_{ik})$, by replacing the terms of *one* column by their derivatives.
[Hint: Use Problem 6 in Chapter 5, §1.]

6. Do Problem 5 for *partials* of f, with $E' = E^n$ (C^n), and for *directionals* $D_{\vec{u}} f$, in *any* normed space E'. (First, prove formulas analogous to Problem 6 in Chapter 5, §1; use Note 3 in §1.) Finally, do it for the *differential*, $df(\vec{p}, \cdot\,)$.

7. In Note 1 of §4, express the matrices in terms of partials (see Theorem 4 in §3). Invent a "variable" notation for such matrices, imitating Jacobians (Corollary 3).

8. (i) Show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \alpha)} = -r^2 \sin \alpha$$

if

$$x = r \cos \theta,$$
$$y = r \sin \theta \, \sin \alpha, \text{ and}$$
$$z = r \cos \alpha$$

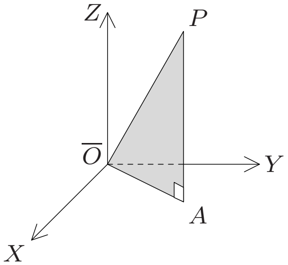

FIGURE 27

(This transformation is passage to polars in E^3; see Figure 27, where $r = OP$, $\sphericalangle XOA = \theta$, and $\sphericalangle AOP = \alpha$.)

(ii) What if $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$ remains unchanged (passage to cylindric coordinates)?

(iii) Same for $x = e^r\cos\theta$, $y = e^r\sin\theta$, and $z = z$.

9. Is $f = (f_1, f_2)\colon E^2 \to E^2$ one-to-one or bijective, and is $J_f \neq 0$, if

(i) $f_1(x,y) = e^x \cos y$ and $f_2(x,y) = e^x \sin y$;

(ii) $f_1(x,y) = x^2 - y^2$ and $f_2(x,y) = 2xy$?

10. Define $f\colon E^3 \to E^3$ (or $C^3 \to C^3$) by

$$f(\vec{x}) = \frac{\vec{x}}{1 + \sum_{k=1}^{3} x_k}$$

on

$$A = \left\{\vec{x} \,\middle|\, \sum_{k=1}^{3} x_k \neq -1\right\}$$

and $f = \vec{0}$ on $-A$. Prove the following.

(i) f is one-to-one on A (find f^{-1}!).

(ii) $J_f(\vec{x}) = \dfrac{1}{\left(1 + \sum_{k=1}^{3} x_k\right)^4}$.

(iii) Describe $-A$ geometrically.

11. Given any sets A, B and maps $f, g\colon A \to E'$, $h\colon E' \to E$, and $k\colon B \to A$, prove that

(i) $(f \pm g) \circ k = f \circ k \pm g \circ k$, and

(ii) $h \circ (f \pm g) = h \circ f + h \circ g$ if h is linear.

Use these *distributive laws* to verify that

$$\phi^{-1} \circ (\theta - \phi) \circ \theta^{-1} = \phi^{-1} - \theta^{-1}$$

in Corollary 3.

[Hint: First verify the *associativity* of mapping composition.]

12. Prove that if $\phi\colon E' \to E$ is linear and one-to-one, so is $\phi^{-1}\colon E'' \to E'$, where $E'' = \phi[E']$.

13. Let $\vec{v}_1, \ldots, \vec{v}_n$ be the column vectors in $\det[\phi]$. Prove that $\det[\phi]$ turns into

(i) $c \cdot \det[\phi]$ if one of the $\vec{v}_k$ is multiplied by a scalar c;

(ii) $-\det[\phi]$, if any two of the $\vec{v}_k$ are interchanged (consider λ in formula (1)).

Furthermore, show that

(iii) $\det[\phi]$ does not change if some $\vec{v}_k$ is replaced by $\vec{v}_k + c\vec{v}_i$ $(i \neq k)$;

(iv) $\det[\phi] = 0$ if some $\vec{v}_k$ is $\vec{0}$, or if two of the $\vec{v}_k$ are the same.

§7. Inverse and Implicit Functions. Open and Closed Maps

I. "If $f \in CD^1$ at $\vec{p}$, then f resembles a *linear map* (namely df) at $\vec{p}$." Pursuing this basic idea, we first make precise our notion of "$f \in CD^1$ at $\vec{p}$."

Definition 1.

A map $f\colon E' \to E$ is *continuously differentiable*, or *of class* CD^1 (written $f \in CD^1$), at $\vec{p}$ iff the following statement is true:

Given any $\varepsilon > 0$, there is $\delta > 0$ such that f is differentiable on the globe $\overline{G} = \overline{G_{\vec{p}}(\delta)}$, with

$$\|df(\vec{x}, \cdot\,) - df(\vec{p}, \cdot\,)\| < \varepsilon \quad \text{for all } \vec{x} \in \overline{G}.$$[1]

By Problem 10 in §5, this definition agrees with Definition 1 of §5, but is no longer limited to the case $E' = E^n$ (C^n). See also Problems 1 and 2 below.

We now obtain the following result.

Theorem 1. *Let E' and E be complete. If $f\colon E' \to E$ is of class CD^1 at $\vec{p}$ and if $df(\vec{p}, \cdot\,)$ is bijective (§6), then f is one-to-one on some globe $\overline{G} = \overline{G_{\vec{p}}(\delta)}$.*

Thus f "locally" resembles $df(\vec{p}, \cdot\,)$ in this respect.

Proof. Set $\phi = df(\vec{p}, \cdot\,)$ and

$$\|\phi^{-1}\| = \frac{1}{\varepsilon}$$

(cf. Theorem 2 of §6).

By Definition 1, fix $\delta > 0$ so that for $\vec{x} \in \overline{G} = \overline{G_{\vec{p}}(\delta)}$.

$$\|df(\vec{x}, \cdot\,) - \phi\| < \frac{1}{2}\varepsilon.$$

Then by Note 5 in §2,

$$(1) \qquad (\forall\, \vec{x} \in \overline{G})\ (\forall\, \vec{u} \in E') \quad |df(\vec{x}, \vec{u}) - \phi(\vec{u})| \le \frac{1}{2}\varepsilon|\vec{u}|.$$

Now fix any $\vec{r}, \vec{s} \in \overline{G}$, $\vec{r} \neq \vec{s}$, and set $\vec{u} = \vec{r} - \vec{s} \neq \vec{0}$. Again, by Note 5 in §2,

$$|\vec{u}| = |\phi^{-1}(\phi(\vec{u}))| \le \|\phi^{-1}\|\,|\phi(\vec{u})| = \frac{1}{\varepsilon}|\phi(\vec{u})|;$$

---

[1] We can always make $\overline{G}$ *closed* by reducing δ.

so

$$0 < \varepsilon|\vec{u}| \le |\phi(\vec{u})|. \tag{2}$$

By convexity, $\overline{G} \supseteq I = L[\vec{s}, \vec{r}]$, so (1) holds for $\vec{x} \in I$, $\vec{x} = \vec{s} + t\vec{u}$, $0 \le t \le 1$. Noting this, set

$$h(t) = f(\vec{s} + t\vec{u}) - t\phi(\vec{u}), \quad t \in E^1.$$

Then for $0 \le t \le 1$,

$$\begin{aligned} h'(t) &= D_{\vec{u}} f(\vec{s} + t\vec{u}) - \phi(\vec{u}) \\ &= df(\vec{s} + t\vec{u}; \vec{u}) - \phi(\vec{u}). \end{aligned}$$

(Verify!) Thus by (1) and (2),

$$\begin{aligned} \sup_{0 \le t \le 1} |h'(t)| &= \sup_{0 \le t \le 1} |df(\vec{s} + t\vec{u}; \vec{u}) - \phi(\vec{u})| \\ &\le \frac{\varepsilon}{2}|\vec{u}| \le \frac{1}{2}|\phi(\vec{u})|. \end{aligned}$$

(Explain!) Now, by Corollary 1 in Chapter 5, §4,

$$|h(1) - h(0)| \le (1 - 0) \cdot \sup_{0 \le t \le 1} |h'(t)| \le \frac{1}{2}|\phi(\vec{u})|.$$

As $h(0) = \vec{s}$ and

$$h(1) = f(\vec{s} + \vec{u}) - \phi(\vec{u}) = f(\vec{r}) - \phi(\vec{u}),$$

we obtain (even if $\vec{r} = \vec{s}$)

$$|f(\vec{r}) - f(\vec{s}) - \phi(\vec{u})| \le \frac{1}{2}|\phi(\vec{u})| \quad (\vec{r}, \vec{s} \in \overline{G},\ \vec{u} = \vec{r} - \vec{s}). \tag{3}$$

But by the triangle law,

$$|\phi(\vec{u})| - |f(\vec{r}) - f(\vec{s})| \le |f(\vec{r}) - f(\vec{s}) - \phi(\vec{u})|.$$

Thus

$$|f(\vec{r}) - f(\vec{s})| \ge \frac{1}{2}|\phi(\vec{u})| \ge \frac{1}{2}\varepsilon|\vec{u}| = \frac{1}{2}\varepsilon|\vec{r} - \vec{s}| \tag{4}$$

by (2).

Hence $f(\vec{r}) \ne f(\vec{s})$ whenever $\vec{r} \ne \vec{s}$ in $\overline{G}$; so f is one-to-one on $\overline{G}$, as claimed. □

Corollary 1. *Under the assumptions of Theorem* 1, *the maps* f *and* f^{-1} (*the inverse of* f restricted to $\overline{G}$) *are uniformly continuous on* $\overline{G}$ *and* $f[\overline{G}]$, *respectively.*

Proof. By (3),

$$\begin{aligned}|f(\vec{r})-f(\vec{s})| &\leq |\phi(\vec{u})|+\frac{1}{2}|\phi(\vec{u})|\\ &\leq |2\,\phi(\vec{u})|\\ &\leq 2\|\phi\|\,|\vec{u}|\\ &= 2\|\phi\|\,|\vec{r}-\vec{s}| \quad (\vec{r},\vec{s}\in\overline{G}).\end{aligned}$$

This implies uniform continuity for f. (Why?)

Next, let $g=f^{-1}$ on $H=f[\overline{G}]$.

If $\vec{x},\vec{y}\in H$, let $\vec{r}=g(\vec{x})$ and $\vec{s}=g(\vec{y})$; so $\vec{r},\vec{s}\in\overline{G}$, with $\vec{x}=f(\vec{r})$ and $\vec{y}=f(\vec{s})$. Hence by (4),

$$|\vec{x}-\vec{y}|\geq\frac{1}{2}\varepsilon|g(\vec{x})-g(\vec{y})|,$$

proving all for g, too. □

Again, f *resembles* ϕ which is uniformly continuous, along with ϕ^{-1}.

II. We introduce the following definition.

Definition 2.

> A map $f\colon(S,\rho)\to(T,\rho')$ is *closed* (*open*) on $D\subseteq S$ iff, for any $X\subseteq D$ the set $f[X]$ is closed (open) *in* T whenever X is so *in* S.

Note that *continuous* maps have such a property for *inverse* images (Problem 15 in Chapter 4, §2).

Corollary 2. *Under the assumptions of Theorem* 1, f *is closed on* $\overline{G}$, *and so the set* $f[\overline{G}]$ *is closed in* E.

Similarly for the map f^{-1} *on* $f[\overline{G}]$.

Proof for $E'=E=E^n\,(C^n)$ (for the general case, see Problem 6)**.** Given any closed $X\subseteq\overline{G}$, we must show that $f[X]$ is closed in E.

Now, as $\overline{G}$ is closed and bounded, it is *compact* (Theorem 4 of Chapter 4, §6).

So also is X (Theorem 1 in Chapter 4, §6), and so is $f[X]$ (Theorem 1 of Chapter 4, §8).

By Theorem 2 in Chapter 4, §6, $f[X]$ is closed, as required. □

For the rest of this section, we shall set $E'=E=E^n\,(C^n)$.

Theorem 2. *If* $E'=E=E^n\,(C^n)$ *in Theorem* 1, *with other assumptions unchanged*, *then* f *is open on the globe* $G=G_{\vec{p}}(\delta)$, *with* δ *sufficiently small.*[2]

We first prove the following lemma.

---

[2] Thus formula (1) still holds for $\varepsilon=1/\|\phi^{-1}\|$, $\phi=df(\vec{p},\cdot)$.

Lemma. *$f[G]$ contains a globe $G_{\vec{q}}(\alpha)$ where $\vec{q} = f(\vec{p})$.*

Proof. Indeed, let

$$\alpha = \frac{1}{4}\varepsilon\delta,$$

where δ and ε are as in the proof of Theorem 1. (We continue the notation and formulas of that proof.)

Fix any $\vec{c} \in G_{\vec{q}}(\alpha)$; so

$$|\vec{c} - \vec{q}| < \alpha = \frac{1}{4}\varepsilon\delta.$$

Set $h = |f - \vec{c}|$ on E'. As f is uniformly continuous on $\overline{G}$, so is h.

Now, $\overline{G}$ is *compact* in E^n (C^n); so Theorem 2(ii) in Chapter 4, §8, yields a point $\vec{r} \in \overline{G}$ such that

$$h(\vec{r}) = \min h[\overline{G}]. \tag{6}$$

We claim that $\vec{r}$ is *in* G (the *interior* of $\overline{G}$).

Otherwise, $|\vec{r} - \vec{p}| = \delta$; for by (4),

$$\begin{aligned} 2\alpha = \frac{1}{2}\varepsilon\delta = \frac{1}{2}\varepsilon|\vec{r} - \vec{p}| &\le |f(\vec{r}) - f(\vec{p})| \\ &\le |f(\vec{r}) - \vec{c}| + |\vec{c} - f(\vec{p})| \\ &= h(\vec{r}) + h(\vec{p}). \end{aligned} \tag{7}$$

But

$$h(\vec{p}) = |\vec{c} - f(\vec{p})| = |\vec{c} - \vec{q}| < \alpha;$$

and so (7) yields

$$h(\vec{p}) < \alpha < h(\vec{r}),$$

contrary to the minimality of $h(\vec{r})$ (see (6)). Thus $|\vec{r} - \vec{p}|$ cannot *equal* δ.

We obtain $|\vec{r} - \vec{p}| < \delta$, so $\vec{r} \in G_{\vec{p}}(\delta) = G$ and $f(\vec{r}) \in f[G]$. We shall now show that $\vec{c} = f(\vec{r})$.

To this end, we set $\vec{v} = \vec{c} - f(\vec{r})$ and prove that $\vec{v} = \vec{0}$. Let

$$\vec{u} = \phi^{-1}(\vec{v}),$$

where

$$\phi = df(\vec{p}, \cdot),$$

as before. Then

$$\vec{v} = \phi(\vec{u}) = df(\vec{p}; \vec{u}).$$

With $\vec{r}$ as above, fix some

$$\vec{s} = \vec{r} + t\vec{u} \quad (0 < t < 1)$$

with t so small that $\vec{s} \in G$ also. Then by formula (3),

$$|f(\vec{s}) - f(\vec{r}) - \phi(t\vec{u})| \leq \frac{1}{2}|t\vec{v}|;$$

also,

$$|f(\vec{r}) - \vec{c} + \phi(t\vec{u})| = (1-t)|\vec{v}| = (1-t)h(\vec{r})$$

by our choice of $\vec{v}, \vec{u}$ and h. Hence by the triangle law,

$$h(\vec{s}) = |f(\vec{s}) - \vec{c}| \leq \left(1 - \frac{1}{2}t\right)h(\vec{r}).$$

(Verify!)

As $0 < t < 1$, this implies $h(\vec{r}) = 0$ (otherwise, $h(\vec{s}) < h(\vec{r})$, violating (6)).

Thus, indeed,

$$|\vec{v}| = |f(\vec{r}) - \vec{c}| = 0,$$

i.e.,

$$\vec{c} = f(\vec{r}) \in f[G] \quad \text{for } \vec{r} \in G.$$

But $\vec{c}$ was an *arbitrary* point of $G_{\vec{q}}(\alpha)$. Hence

$$G_{\vec{q}}(\alpha) \subseteq f[G],$$

proving the lemma. □

Proof of Theorem 2. The lemma shows that $f(\vec{p})$ is in the *interior* of $f[G]$ if $\vec{p}$, f, $df(\vec{p}, \cdot\,)$, and δ are as in Theorem 1.

But Definition 1 implies that here $f \in CD^1$ on *all* of G (see Problem 1).

Also, $df(\vec{x}, \cdot\,)$ is *bijective* for *any* $\vec{x} \in G$ by our choice of G and Theorems 1 and 2 in §6.

Thus f maps *all* $\vec{x} \in G$ onto interior points of $f[G]$; i.e., f maps any open set $X \subseteq G$ onto an *open* $f[X]$, as required. □

Note 1. A map

$$f\colon (S, \rho) \underset{\text{onto}}{\longleftrightarrow} (T, \rho')$$

is both open and closed ("*clopen*") iff f^{-1} is continuous—see Problem 15(iv)(v) in Chapter 4, §2, interchanging f and f^{-1}.

Thus $\phi = df(\vec{p}, \cdot\,)$ in Theorem 1 is "clopen" on all of E'.

Again, f *locally* resembles $df(\vec{p}, \cdot\,)$.

III. The Inverse Function Theorem. We now further pursue these ideas.

Theorem 3 (inverse functions). *Under the assumptions of Theorem 2, let g be the inverse of f_G (f restricted to $G = G_{\vec{p}}(\delta)$).*

Then $g \in CD^1$ on $f[G]$ and $dg(\vec{y}, \cdot\,)$ is the inverse of $df(\vec{x}, \cdot\,)$ whenever $\vec{x} = g(\vec{y})$, $\vec{x} \in G$.

Briefly: "*The differential of the inverse is the inverse of the differential.*"

Proof. Fix any $\vec{y} \in f[G]$ and $\vec{x} = g(\vec{y})$; so $\vec{y} = f(\vec{x})$ and $\vec{x} \in G$. Let $U = df(\vec{x}, \cdot\,)$.

As noted above, U is *bijective* for every $\vec{x} \in G$ by Theorems 1 and 2 in §6; so we may set $V = U^{-1}$. We must show that $V = dg(\vec{y}, \cdot\,)$.

To do this, give $\vec{y}$ an arbitrary (variable) increment $\Delta\vec{y}$, so small that $\vec{y}+\Delta\vec{y}$ stays in $f[G]$ (an *open* set by Theorem 2).

As g and f_G are one-to-one, $\Delta\vec{y}$ *uniquely* determines

$$\Delta\vec{x} = g(\vec{y} + \Delta\vec{y}) - g(\vec{y}) = \vec{t},$$

and vice versa:

$$\Delta\vec{y} = f(\vec{x} + \vec{t}) - f(\vec{x}).$$

Here $\Delta\vec{y}$ and $\vec{t}$ are the *mutually corresponding* increments of $\vec{y} = f(\vec{x})$ and $\vec{x} = g(\vec{y})$. By continuity, $\vec{y} \to \vec{0}$ iff $\vec{t} \to \vec{0}$.[3]

As $U = df(\vec{x}, \cdot\,)$,

$$\lim_{\vec{t} \to \vec{0}} \frac{1}{|\vec{t}|} |f(\vec{x} + \vec{t}) - f(\vec{t}) - U(\vec{t})| = 0,$$

or

$$\lim_{\vec{t} \to \vec{0}} \frac{1}{|\vec{t}|} |F(\vec{t})| = 0, \tag{8}$$

where

$$F(\vec{t}) = f(\vec{x} + \vec{t}) - f(\vec{t}) - U(\vec{t}). \tag{9}$$

As $V = U^{-1}$, we have

$$V(U(\vec{t})) = \vec{t} = g(\vec{y} + \Delta\vec{y}) - g(\vec{y}).$$

So from (9),

$$\begin{aligned} V(F(\vec{t})) &= V(\Delta\vec{y}) - \vec{t} \\ &= V(\Delta\vec{y}) - [g(\vec{y} + \Delta\vec{y}) - g(\vec{y})]; \end{aligned}$$

that is,

$$\frac{1}{|\Delta\vec{y}|} |g(\vec{y} + \Delta\vec{y}) - g(\vec{y}) - V(\Delta\vec{y})| = \frac{|V(F(\vec{t}))|}{|\Delta\vec{y}|}, \quad \Delta\vec{y} \neq \vec{0}. \tag{10}$$

Now, formula (4), with $\vec{r} = \vec{x}$, $\vec{s} = \vec{x} + \vec{t}$, and $\vec{u} = \vec{t}$, shows that

$$|f(\vec{x} + \vec{t}) - f(\vec{x})| \geq \frac{1}{2}\varepsilon|\vec{t}|;$$

---

[3] This change of variables is *admissible* as the map $\vec{t} \longleftrightarrow \Delta\vec{y}$ is one-to-one (Corollary 2 in Chapter 4, §2).

i.e., $|\Delta\vec{y}| \geq \frac{1}{2}\varepsilon|\vec{t}|$. Hence by (8),

$$\frac{|V(F(\vec{t}))|}{|\Delta\vec{y}|} \leq \frac{|V(F(\vec{t})|}{\frac{1}{2}\varepsilon|\vec{t}|} = \frac{2}{\varepsilon}\left|V\left(\frac{1}{|\vec{t}|}F(\vec{t})\right)\right| \leq \frac{2}{\varepsilon}\,\|V\|\,\frac{1}{|\vec{t}|}|F(\vec{t})| \to 0 \text{ as } \vec{t} \to \vec{0}.$$

Since $\vec{t} \to \vec{0}$ as $\Delta\vec{y} \to \vec{0}$ (change of variables!), the expression (10) tends to 0 as $\Delta\vec{y} \to \vec{0}$.

By definition, then, g is differentiable at $\vec{y}$, with $dg(\vec{y}, \cdot\,) = V = U^{-1}$.

Moreover, Corollary 3 in §6, applies here. Thus

$$(\forall\,\delta' > 0)\ (\exists\,\delta'' > 0) \quad \|U - W\| < \delta'' \Rightarrow \|U^{-1} - W^{-1}\| < \delta'.$$

Taking here $U^{-1} = dg(\vec{y}, \cdot\,)$ and $W^{-1} = dg(\vec{y} + \Delta\vec{y})$, we see that $g \in CD^1$ near $\vec{y}$. This completes the proof. $\square$

Note 2. If $E' = E = E^n$ (C^n), the bijectivity of $\phi = df(\vec{p}, \cdot\,)$ is equivalent to

$$\det[\phi] = \det[f'(\vec{p})] \neq 0$$

(Theorem 1 of §6).

In this case, the fact that f is one-to-one on $G = G_{\vec{p}}(\delta)$ means, *componentwise* (see Note 3 in §6), that the system of n equations

$$f_i(\vec{x}) = f(x_1, \ldots, x_n) = y_i, \quad i = 1, \ldots, n,$$

has a unique solution for the n unknowns x_k as long as

$$(y_1, \ldots, y_n) = \vec{y} \in f[G].$$

Theorem 3 shows that this solution has the form

$$x_k = g_k(\vec{y}), \quad k = 1, \ldots, n,$$

where the g_k are of class CD^1 on $f[G]$ provided the f_i are of class CD^1 near $\vec{p}$ and $\det\,[f'(\vec{p})] \neq 0$. Here

$$\det[f'(\vec{p})] = J_f(\vec{p}),$$

as in §6.

Thus again f "*locally*" resembles a linear map, $\phi = df(\vec{p}, \cdot\,)$.

IV. The Implicit Function Theorem. Generalizing, we now ask, what about solving n *equations in* $n + m$ unknowns $x_1, \ldots, x_n, y_1, \ldots, y_m$? Say, we want to solve

$$f_k(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0, \quad k = 1, 2, \ldots, n, \tag{11}$$

for the first n unknowns (or variables) x_k, thus expressing them as

$$x_k = H_k(y_1, \ldots, y_m), \quad k = 1, \ldots, n,$$

with $H_k\colon E^m \to E^1$ or $H_k\colon C^m \to C$.

Let us set $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_m)$, and

$$(\vec{x}, \vec{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$$

so that $(\vec{x}, \vec{y}) \in E^{n+m}\,(C^{n+m})$.

Thus the system of equations (11) simplifies to

$$f_k(\vec{x}, \vec{y}) = 0, \quad k = 1, \ldots, n,$$

or

$$f(\vec{x}, \vec{y}) = \vec{0},$$

where $f = (f_1, \ldots, f_n)$ *is a map of* $E^{n+m}\,(C^{n+m})$ *into* $E^n\,(C^n)$; f is a function of $n + m$ *variables*, but it has n *components* f_k; i.e.,

$$f(\vec{x}, \vec{y}) = f(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

is a vector *in* $E^n\,(C^n)$.

Theorem 4 (implicit functions)**.** *Let* $E' = E^{n+m}\,(C^{n+m})$, $E = E^n\,(C^n)$, *and let* $f\colon E' \to E$ *be of class* CD^1 *near*

$$(\vec{p}, \vec{q}) = (p_1, \ldots, p_n, q_1, \ldots, q_m), \quad \vec{p} \in E^n\,(C^n),\ \vec{q} \in E^m\,(C^m).$$

Let $[\phi]$ *be the* $n \times n$ *matrix*

$$(D_j f_k(\vec{p}, \vec{q})), \quad j, k = 1, \ldots, n.$$

If $\det[\phi] \neq 0$ *and if* $f(\vec{p}, \vec{q}) = \vec{0}$, *then there are open sets*

$$P \subseteq E^n\,(C^n) \text{ and } Q \subseteq E^m\,(C^m),$$

with $\vec{p} \in P$ *and* $\vec{q} \in Q$, *for which there is a unique map*

$$H\colon Q \to P$$

with

$$f(H(\vec{y}), \vec{y}) = \vec{0}$$

for all $\vec{y} \in Q$; *furthermore,* $H \in CD^1$ *on* Q.

Thus $\vec{x} = H(\vec{y})$ is a solution of (11) in vector form.

Proof. With the above notation, set

$$F(\vec{x}, \vec{y}) = (f(\vec{x}, \vec{y}), \vec{y}), \quad F\colon E' \to E'.$$

Then

$$F(\vec{p}, \vec{q}) = (f(\vec{p}, \vec{q}), \vec{q}) = (\vec{0}, \vec{q}),$$

since $f(\vec{p}, \vec{q}) = \vec{0}$.

As $f \in CD^1$ near $(\vec{p}, \vec{q})$, so is F (verify *componentwise* via Problem 9(ii) in §3 and Definition 1 of §5).

By Theorem 4, §3, $\det[F'(\vec{p},\vec{q})] = \det[\phi] \neq 0$ (explain!).

Thus Theorem 1 above shows that F is one-to-one on some globe G about $(\vec{p},\vec{q})$.

Clearly G contains an open *interval* about $(\vec{p},\vec{q})$. We denote it by $P \times Q$ where $\vec{p} \in P$, $\vec{q} \in Q$; P is open in E^n (C^n) and Q is open in E^m (C^m).[4]

By Theorem 3, $F_{P\times Q}$ (F restricted to $P \times Q$) has an inverse

$$g\colon A \underset{\text{onto}}{\longleftrightarrow} P \times Q,$$

where $A = F[P \times Q]$ is open *in* E' (Theorem 2), and $g \in CD^1$ on A. Let the map $u = (g_1, \ldots, g_n)$ comprise the *first* n components of g (exactly as f comprises the first n components of F).

Then

$$g(\vec{x},\vec{y}) = (u(\vec{x},\vec{y}),\vec{y})$$

exactly as $F(\vec{x},\vec{y}) = (f(\vec{x},\vec{y}),\vec{y})$. Also, $u\colon A \to P$ is of class CD^1 on A, as g is (explain!).

Now set

$$H(\vec{y}) = u(\vec{0},\vec{y});$$

here $\vec{y} \in Q$, while

$$(\vec{0},\vec{y}) \in A = F[P \times Q],$$

for F *preserves* $\vec{y}$ (the *last* m coordinates). Also set

$$\alpha(\vec{x},\vec{y}) = \vec{x}.$$

Then $f = \alpha \circ F$ (why?), and

$$f(H(\vec{y}),\vec{y}) = f(u(\vec{0},\vec{y}),\vec{y}) = f(g(\vec{0},\vec{y})) = \alpha(F(g(\vec{0},\vec{y})) = \alpha(\vec{0},\vec{y}) = \vec{0}$$

by our choice of α and g (inverse to F). Thus

$$f(H(\vec{y}),\vec{y}) = \vec{0}, \quad \vec{y} \in Q,$$

as desired.

Moreover, as $H(\vec{y}) = u(\vec{0},\vec{y})$, we have

$$\frac{\partial}{\partial y_i} H(\vec{y}) = \frac{\partial}{\partial y_i} u(\vec{0},\vec{y}), \quad \vec{y} \in Q, i \leq m.$$

As $u \in CD^1$, all $\partial u/\partial y_i$ are continuous (Definition 1 in §5); hence so are the $\partial H/\partial y_i$. Thus by Theorem 3 in §3, $H \in CD^1$ on Q.

---

[4] This can be made more precise using the theory of *product spaces* (Chapter 4, *§11).

Finally, H is *unique* for the given P, Q; for

$$\begin{aligned} f(\vec{x},\vec{y}) = \vec{0} &\Longrightarrow (f(\vec{x},\vec{y}),\vec{y}) = (\vec{0},\vec{y}) \\ &\Longrightarrow F(\vec{x},\vec{y}) = (\vec{0},\vec{y}) \\ &\Longrightarrow g(F(\vec{x},\vec{y})) = g(\vec{0},\vec{y}) \\ &\Longrightarrow (\vec{x},\vec{y}) = g(\vec{0},\vec{y}) = (u(\vec{0},\vec{y}),\vec{y}) \\ &\Longrightarrow \vec{x} = u(\vec{0},\vec{y}) = H(\vec{y}). \end{aligned}$$

Thus $f(\vec{x},\vec{y}) = \vec{0}$ *implies* $\vec{x} = H(\vec{y})$; so $H(\vec{y})$ is the *only* solution for $\vec{x}$. □

Note 3. H is said to be *implicitly* defined by the equation $f(\vec{x},\vec{y}) = \vec{0}$. In this sense we say that $H(\vec{y})$ is an *implicit function*, given by $f(\vec{x},\vec{y}) = \vec{0}$.

Similarly, under suitable assumptions, $f(\vec{x},\vec{y}) = \vec{0}$ defines $\vec{y}$ as a function of $\vec{x}$.

Note 4. While H is unique for a *given* neighborhood $P \times Q$ of $(\vec{p},\vec{q})$, *another* implicit function may result if $P \times Q$ or $(\vec{p},\vec{q})$ is *changed*.

For example, let

$$f(x,y) = x^2 + y^2 - 25$$

(a polynomial; hence $f \in CD^1$ on all of E^2). Geometrically, $x^2 + y^2 - 25 = 0$ describes a circle.

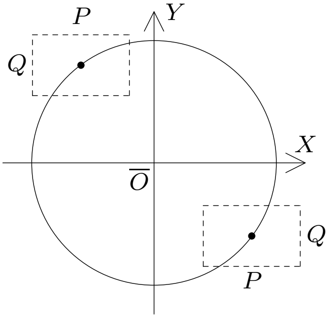

FIGURE 28

Solving for x, we get $x = \pm\sqrt{25 - y^2}$. Thus we have *two* functions:

$$H_1(y) = +\sqrt{25 - y^2}$$

and

$$H_2(y) = -\sqrt{25 - y^2}.$$

If $P \times Q$ is in the upper part of the circle, the resulting function is H_1. Otherwise, it is H_2. See Figure 28.

V. Implicit Differentiation. Theorem 4 only states the *existence* (and uniqueness) of a solution, but does not show how to *find* it, in general.

The knowledge itself that $H \in CD^1$ *exists*, however, enables us to use its derivative or partials and compute it by *implicit differentiation*, known from calculus.[5]

---

[5] For more on implicit differentiation, see §10.

Examples.

(a) Let $f(x, y) = x^2 + y^2 - 25 = 0$, as above.

This time treating y as an implicit function of $x, y = H(x)$, and writing y' for $H'(x)$, we differentiate both sides of $x^2 + y^2 - 25 = 0$ with respect to x, using the chain rule for the term $y^2 = [H(x)]^2$.

This yields $2x + 2yy' = 0$, whence $y' = -x/y$.

Actually (see Note 4), *two* functions are involved: $y = \pm\sqrt{25 - x^2}$; but both satisfy $x^2 + y^2 - 25 = 0$; so the result $y' = -x/y$ applies to both.

Of course, this method is possible only if the derivative y' is *known* to exist. This is why Theorem 4 is important.

(b) Let

$$f(x, y, z) = x^2 + y^2 + z^2 - 1 = 0, \quad x, y, z \in E^1.$$

Again f satisfies Theorem 4 for suitable x, y, and z.

Setting $z = H(x, y)$, differentiate the equation $f(x, y, z) = 0$ *partially* with respect to x and y. From the resulting two equations, obtain $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Problems on Inverse and Implicit Functions, Open and Closed Maps

1. Discuss: In Definition 1, $\overline{G}$ can equivalently be replaced by $G = G_{\vec{p}}(\delta)$ (an *open* globe).

2. Prove that if the set D is open (closed) in (S, ρ), then the map $f\colon S \to T$ is open (closed, respectively) on D iff f_D (f restricted to D) has this property *as a map of D into $f[D]$*.

[Hint: Use Theorem 4 in Chapter 3, §12.]

3. Complete the missing details in the proofs of Theorems 1–4.

3′ Verify footnotes 2 and 3.

4. Show that a map $f\colon E' \to E$ may fail to be one-to-one on *all* of E' even if f satisfies Theorem 1 near *every* $\vec{p} \in E'$. Nonetheless, show that this cannot occur if $E' = E = E^1$.

[Hints: For the first part, take $E' = C$, $f(x+iy) = e^x(\cos y + i \sin y)$. For the second, use Theorem 1 in Chapter 5, §2.]

4′. (i) For maps $f\colon E^1 \to E^1$, prove that the existence of a bijective $df(p, \cdot\,)$ is equivalent to $f'(p) \neq 0$.

(ii) Let

$$f(x) = x + x^2 \sin \frac{1}{x}, \quad f(0) = 0.$$

Show that $f'(0) \neq 0$, and $f \in CD^1$ near any $p \neq 0$; yet f is *not one-to-one near* 0. What is wrong?

5. Show that a map $f\colon E^n\,(C^n) \to E^n\,(C^n)$, $f \in CD^1$, may be bijective even if $\det[f'(\vec{p})] = 0$ at some $\vec{p}$, but then f^{-1} cannot be differentiable at $\vec{q} = f(\vec{p})$.

[Hint: For the first clause, take $f(x) = x^3$, $p = 0$; for the second, note that if f^{-1} is differentiable at $\vec{q}$, then Note 2 in §4 implies that $\det[df(\vec{p}, \cdot)] \cdot \det[df^{-1}(\vec{q}, \cdot)] = 1 \neq 0$, since $f \circ f^{-1}$ is the identity map.]

6. Prove Corollary 2 for the general case of complete E' and E.

[Outline: Given a closed $X \subseteq \overline{G}$, take any *convergent* sequence $\{\vec{y}_n\} \subseteq f[X]$. By Problem 8 in Chapter 4, §8, $f^{-1}(\vec{y}_n) = \vec{x}_n$ is a *Cauchy sequence* in X (why?). By the completeness of E', $(\exists\, \vec{x} \in X)\ \vec{x}_n \to \vec{x}$ (Theorem 4 of Chapter 3, §16). Infer that $\lim \vec{y}_n = f(\vec{x}) \in f[X]$, so $f[X]$ is closed.]

7. Prove that "the composite of two open (closed) maps is open (closed)." State the theorem precisely. Prove it also for the uniform *Lipschitz* property.

8. Prove in detail that $f\colon (S, \rho) \to (T, \rho')$ is open on $D \subseteq S$ iff f maps the interior of D into that of $f[D]$; that is, $f[D^0] \subseteq (f[D])^0$.

9. Verify by examples that f may be:

(i) closed but not open;

(ii) open but not closed.

[Hints: (i) Consider $f = $ constant. (ii) Define $f\colon E^2 \to E^1$ by $f(x, y) = x$ and let

$$D = \left\{ (x, y) \in E^2 \,\middle|\, y = \frac{1}{x},\ x > 0 \right\};$$

use Theorem 4(iii) in Chapter 3, §16 and continuity to show that D is closed in E^2, but $f[D] = (0, +\infty)$ is not closed *in* E^1. However, f is open on all of E^2 by Problem 8. (Verify!)]

10. Continuing Problem 9(ii), define $f\colon E^n \to E^1$ (or $C^n \to C$) by $f(\vec{x}) = x_k$ for a fixed $k \leq n$ (the "*kth projection map*"). Show that f is open, but not closed, on $E^n\,(C^n)$.

11. (i) In Example (a), take $(p, q) = (5, 0)$ or $(-5, 0)$. Are the conditions of Theorem 4 satisfied? Do the conclusions hold?

(ii) Verify Example (b).

12. (i) Treating z as a function of x and y, given implicitly by

$$f(x, y, z) = z^3 + xz^2 - yz = 0, \quad f\colon E^3 \to E^1,$$

discuss the choices of P and Q that satisfy Theorem 4. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(ii) Do the same for $f(x, y, z) = e^{xyz} - 1 = 0$.

13. Given $f\colon E^n\,(C^n) \to E^m\,(C^m)$, $n > m$, prove that if $f \in CD^1$ on a globe G, f cannot be one-to-one.

[Hint for $f\colon E^2 \to E^1$: If, say, $D_1 f \neq 0$ on G, set $F(x, y) = (f(x, y), y)$.]

14. Suppose that f satisfies Theorem 1 for every $\vec{p}$ in an open set $A \subseteq E'$, *and is one-to-one on* A (cf. Problem 4). Let $g = f_A^{-1}$ (restrict f to A and take its inverse). Show that f and g are open and of class CD^1 on A and $f[A]$, respectively.

15. Given $\vec{v} \in E$ and a scalar $c \neq 0$, define $T_{\vec{v}} \colon E \to E$ ("translation by $\vec{v}$") and $M_c \colon E \to E$ ("dilation by c"), by setting

$$T_{\vec{v}}(\vec{x}) = \vec{x} + \vec{v} \text{ and } M_c(\vec{x}) = c\vec{x}.$$

Prove the following.

(i) $T_{\vec{v}}$ and $T_{\vec{v}}^{-1} (= T_{-\vec{v}})$ are bijective, continuous, and "clopen" on E; so also are M_c and $M_c^{-1} (= M_{1/c})$.

(ii) Similarly for the Lipschitz property on E.

(iii) If $G = G_{\vec{q}}(\delta) \subset E$, then $T_{\vec{v}}[G] = G_{\vec{q}+\vec{v}}(\delta)$, and $M_c[G] = G_{c\vec{q}}(|c\delta|)$.

(iv) If $f \colon E' \to E$ is linear, and $\vec{v} = f(\vec{p})$ for some $\vec{p} \in E'$, then $T_{\vec{v}} \circ f = f \circ T'_{\vec{p}}$ and $M_c \circ f = f \circ M'_c$, where $T'_{\vec{p}}$ and M'_c are the corresponding maps *on* E'. If, further, f is continuous at $\vec{p}$, it is continuous on *all* of E'.

[Hint for (iv): Fix any $\vec{x} \in E'$. Set $\vec{v} = f(\vec{x} - \vec{p})$, $g = T_{\vec{v}} \circ f \circ T'_{\vec{p}-\vec{x}}$. Verify that $g = f$, $T'_{\vec{p}-\vec{x}}(\vec{x}) = \vec{p}$, and g is continuous *at* $\vec{x}$.]

16. Show that if $f \colon E' \to E$ is linear and if $f[G^*]$ is open in E for some $G^* = G_{\vec{p}}(\delta) \subseteq E'$, then

(i) f is open on all of E';

(ii) f is *onto* E.

[Hints: (i) By Problem 8, it suffices to show that the set $f[G]$ is open, for any *globe* G (why?). First take $G = G_{\vec{0}}(\delta)$. Then use Problems 7 and 15(i)–(iv), with suitable $\vec{v}$ and c.

(ii) To prove $E = f[E']$, fix any $\vec{y} \in E$. As $f = G_{\vec{0}}(\delta)$ is open, it contains a globe $G' = G_{\vec{0}}(r)$. For small c, $c\vec{y} \in G' \subseteq f[E']$. Hence $\vec{y} \in f[E']$ (Problem 10 in §2).]

17. Continuing Problem 16, show that if f is also one-to-one on G^*, then

$$f \colon E' \underset{\text{onto}}{\longleftrightarrow} E,$$

$f \in L(E', E)$, $f^{-1} \in L(E, E')$, f is *clopen* on E', and f^{-1} is so on E.

[Hints: To prove that f is one-to-one *on* E', let $f(\vec{x}) = f(\vec{x}\,') = \vec{y}$ for some $\vec{x}, \vec{x}\,' \in E'$. Show that

$$(\exists\, c,\, \varepsilon > 0) \quad c\vec{y} \in G_{\vec{0}}(\varepsilon) \subseteq f[G_{\vec{0}}(\delta)] \text{ and } f(c\vec{x} + \vec{p}) = f(c\vec{x}\,' + \vec{p}) \in f[G_{\vec{p}}(\delta)] = f[G^*].$$

Deduce that $c\vec{x} + \vec{p} = c\vec{x}\,' + \vec{p}$ and $\vec{x} = \vec{x}\,'$. Then use Problem 15(v) in Chapter 4, §2, and Note 1.]

18. A map

$$f \colon (S, \rho) \underset{\text{onto}}{\longleftrightarrow} (T, \rho')$$

is said to be *bicontinuous*, or a *homeomorphism*, (from S onto T) iff both f and f^{-1} are continuous. Assuming this, prove the following.

(i) $x_n \to p$ in S iff $f(x_n) \to f(p)$ in T;

(ii) A is closed (open, compact, perfect) in S iff $f[A]$ is so in T;

(iii) $B = \overline{A}$ in S iff $f[B] = \overline{f[A]}$ in T;

(iv) $B = A^0$ in S iff $f[B] = (f[A])^0$ in T;

(v) A is dense in B (i.e., $A \subseteq B \subseteq \overline{A} \subseteq S$) in (S, ρ) iff $f[A]$ is dense in $f[B] \subseteq (T, \rho')$.

[Hint: Use Theorem 1 of Chapter 4, §2, and Theorem 4 in Chapter 3, §16, for closed sets; see also Note 1.]

19. Given $A, B \subseteq E$, $\vec{v} \in E$ and a scalar c, set

$$A + \vec{v} = \{\, \vec{x} + \vec{v} \mid \vec{x} \in A \,\} \text{ and } cA = \{\, c\vec{x} \mid \vec{x} \in A \,\}.$$

Assuming $c \neq 0$, prove that

(i) A is closed (open, compact, perfect) in E iff $cA + \vec{v}$ is;

(ii) $B = \overline{A}$ iff $cB + \vec{v} = \overline{cA + \vec{v}}$;

(iii) $B = A^0$ iff $cB + \vec{v} = (cA + \vec{v})^0$;

(iv) A is dense in B iff $cA + \vec{v}$ is dense in $cB + \vec{v}$.

[Hint: Apply Problem 18 to the maps $T_{\vec{v}}$ and M_c of Problem 15, noting that $A + \vec{v} = T_{\vec{v}}[A]$ and $cA = M_c[A]$.]

20. Prove Theorem 2, for a *reduced* δ, assuming that only *one* of E' and E is E^n (C^n), and the other is just complete.

[Hint: If, say, $E = E^n$ (C^n), then $f[\overline{G}]$ is compact (being closed and bounded), and so is $\overline{G} = f^{-1}[f[\overline{G}]]$. (Why?) Thus the Lemma works out as before, i.e., $f[G] \supseteq G_{\vec{q}}(\alpha)$.

Now use the continuity of f to obtain a globe $G' = G_{\vec{p}}(\delta') \subseteq G$ such that $f[G'] \subseteq G_{\vec{q}}(\alpha)$. Let $g = f_G^{-1}$, further restricted to $G_{\vec{q}}(\alpha)$. Apply Problem 15(v) in Chapter 4, §2, to g, with $S = G_{\vec{q}}(\alpha)$, $T = E'$.]

*§8. Baire Categories. More on Linear Maps

We pause to outline the theory of so-called sets of Category I or Category II, as introduced by Baire. It is one of the most powerful tools in higher analysis. Below, (S, ρ) is a metric space.

Definition 1.

A set $A \subseteq (S, \rho)$ is said to be *nowhere dense* (in S) iff its closure $\overline{A}$ has no interior points (i.e., contains no globes): $(\overline{A})^0 = \emptyset$.

Equivalently, the set A is nowhere dense iff *every open set $G^* \neq \emptyset$ in S contains a globe $\overline{G}$ disjoint from A.* (Why?)

Definition 2.

A set $A \subseteq (S, \rho)$ is *meagre*, or *of Category* I (in S), iff

$$A = \bigcup_{n=1}^{\infty} A_n,$$

for some sequence of nowhere dense sets A_n.

Otherwise, A is said to be *nonmeagre* or *of Category* II.

A is *residual* iff $-A$ is meagre, but A is not.

Examples.

(a) $\emptyset$ is nowhere dense.

(b) Any finite set in a normed space E is nowhere dense.

(c) The set N of all naturals in E^1 is nowhere dense.

(d) So also is *Cantor's set* P (Problem 17 in Chapter 3, §14); indeed, P is closed ($P = \overline{P}$) and has no interior points (verify!), so $(\overline{P})^0 = P^0 = \emptyset$.

(e) The set R of all rationals in E^1 is *meagre*; for it is *countable* (see Chapter 1, §9), hence a countable union of nowhere dense singletons $\{r_n\}$, $r_n \in R$. But R is not nowhere dense; it is even *dense* in E^1, since $\overline{R} = E^1$ (see Definition 2, in Chapter 3, §14). Thus *a meagre set need not be nowhere dense.* (But all nowhere dense sets are meagre—why?)

Examples (c) and (d) show that a nowhere dense set may be infinite (even uncountable). Yet, sometimes nowhere dense sets are treated as "small" or "negligible," in comparison with other sets. Most important is the following theorem.

Theorem 1 (Baire)**.** *In a complete metric space* (S, ρ), *every open set* $G^* \neq \emptyset$ *is nonmeagre. Hence the entire space* S *is residual.*

Proof. Seeking a contradiction, suppose G^* is meagre, i.e.,

$$G^* = \bigcup_{n=1}^{\infty} A_n$$

for some nowhere dense sets A_n. Now, as A_1 is nowhere dense, G^* contains a closed globe

$$\overline{G}_1 = \overline{G_{x_1}(\delta_1)} \subseteq -A_1.$$

Again, as A_2 is nowhere dense, G_1 contains a globe

$$\overline{G}_2 = \overline{G_{x_2}(\delta_2)} \subseteq -A_2, \quad \text{with } 0 < \delta_2 \leq \frac{1}{2}\delta_1.$$

By induction, we obtain a *contracting* sequence of closed globes

$$\overline{G}_n = \overline{G_{x_n}(\delta_n)}, \quad \text{with } 0 < \delta_n \leq \frac{1}{2^n}\delta_1 \to 0.$$

As S is complete, so are the $\overline{G}_n$ (Theorem 5 in Chapter 3, §17). Thus, by Cantor's theorem (Theorem 5 of Chapter 4, §6), there is

$$p \in \bigcap_{n=1}^{\infty} \overline{G}_n.$$

As $G^* \supseteq \overline{G}_n$, we have $p \in G^*$. But, as $\overline{G}_n \subseteq -A_n$, we also have $(\forall\, n)\ p \notin A_n$; hence

$$p \notin \bigcup_{n=1}^{\infty} A_n = G^*$$

(the desired contradiction!). $\square$

We shall need a lemma based on Problems 15 and 19 in §7. (Review them!)

Lemma. *Let $f \in L(E', E)$, E' complete. Let $G = G_{\vec{0}}(1)$ be the unit globe in E'. If $\overline{f[G]}$ (closure of $f[G]$ in E) contains a globe $G_0 = G_0(r) \subset E$, then $G_0 \subseteq f[G]$.*

Note. Recall that we "arrow" only vectors from E' (e.g., $\vec{0}$), but not those from E (e.g., 0).

Proof of lemma. Let $A = f[G] \cap G_0 \subseteq G_0$. We claim that A is *dense* in G_0; i.e., $G_0 \subseteq \overline{A}$. Indeed, by assumption, any $q \in G_0$ is in $\overline{f[G]}$. Thus by Theorem 3 in Chapter 3, §16, any G_q meets $f[G] \cap G_0 = A$ if $q \in G_0$. Hence

$$(\forall\, q \in G_0) \quad q \in \overline{A},$$

i.e., $G_0 \subseteq \overline{A}$, as claimed.

Now fix any $q_0 \in G_0 = G_0(r)$ and a real c $(0 < c < 1)$. As A is dense in G_0,

$$A \cap G_{q_0}(cr) \neq \emptyset;$$

so let $q_1 \in A \cap G_{q_0}(cr) \subseteq f[G]$. Then

$$|q_1 - q_0| < cr, \quad q_0 \in G_{q_1}(cr).$$

As $q_1 \in f[G]$, we can fix some $\vec{p}_1 \in G = G_0(1)$, with $f(\vec{p}_1) = q_1$. Also, by Problems 19(iv) and 15(iii) in §7, $cA + q_1$ is dense in $cG_0 + q_1 = G_{q_1}(cr)$. But $q_0 \in G_{q_1}(cr)$. Thus

$$G_{q_0}(c^2r) \cap (cA + q_1) \neq \emptyset;$$

so let $q_2 \in G_{q_0}(c^2r) \cap (cA + q_1)$, so $q_0 \in G_{q_2}(c^2r)$, etc.

Inductively, we fix for each $n > 1$ some $q_n \in G_{q_0}(c^n r)$, with

$$q_n \in c^{n-1}A + q_{n-1},$$

i.e.,

$$q_n - q_{n-1} \in c^{n-1}A.$$

As $A \subseteq f[G_0(1)]$, linearity yields

$$q_n - q_{n-1} \in f[c^{n-1}G_0(1)] = f[G_0(c^{n-1})], \quad n > 1.$$

Thus for each $n > 1$, there is $\vec{p}_n \in G_0(c^{n-1})$, (i.e., $|\vec{p}_n| < c^{n-1}$) such that $f(\vec{p}_n) = q_n - q_{n-1}$. Now, as $|\vec{p}_n| < c^{n-1}$ and $0 < c < 1$,

$$\sum_1^\infty |\vec{p}_n| < +\infty;$$

so by the completeness of E', $\sum \vec{p}_n$ converges in E' (Theorem 1 in Chapter 4, §13). Let $\vec{p} = \sum_{k=1}^\infty \vec{p}_k$; then

$$\begin{aligned} f(\vec{p}) &= f\biggl(\lim_{n\to\infty} \sum_{k=1}^n \vec{p}_k\biggr) = \lim_{n\to\infty} f\biggl(\sum_{k=1}^n \vec{p}_k\biggr) \\ &= \lim_{n\to\infty} \sum_{k=1}^n f(\vec{p}_k) \quad \text{for } f \in L(E', E). \end{aligned}$$

But $f(\vec{p}_k) = q_k - q_{k-1}$ $(k > 1)$, and $f(\vec{p}_1) = q_1$; so

$$\sum_{k=1}^n f(\vec{p}_k) = q_1 + \sum_{k=2}^n (q_k - q_{k-1}) = q_n.$$

Thus

$$f(\vec{p}) = \lim_{n\to\infty} \sum_{k=1}^n f(\vec{p}_k) = \lim_{n\to\infty} q_n = q_0.^{[1]}$$

Moreover, $|\vec{p}_k| < c^{k-1}$ $(k \geq 1)$. Thus

$$|\vec{p}| \leq \sum_{k=1}^\infty |\vec{p}_k| < \sum_{k=1}^\infty c^{k-1} = \frac{1}{1-c};$$

i.e.,

$$\vec{p} \in G_{\vec{0}}\Bigl(\frac{1}{1-c}\Bigr).$$

But $q_0 = f(\vec{p})$; so

$$q_0 \in f\Bigl[G_{\vec{0}}\Bigl(\frac{1}{1-c}\Bigr)\Bigr].$$

---

[1] Note that $q_n \to q_0$, since $q_n \in G_{q_0}(c^n r)$ implies $|q_n - q_0| < c^n r \to 0$, as $0 < c < 1$.

As $q_0 \in G_0(r)$ was *arbitrary*, we have

$$G_0(r) \subseteq f\Big[G_0\Big(\frac{1}{1-c}\Big)\Big],$$

or by linearity,

$$G_0(r(1-c)) \subseteq f[G_0(1)] = f[G].$$

This holds for any $c \in (0,1)$. Hence

$$f[G] \supseteq \bigcup_{0<c<1} G_0(r(1-c)) = G_0(r). \quad \text{(Verify!)}$$

Thus all is proved. $\square$

We can now establish an important result due to S. Banach.

Theorem 2 (Banach). *Let $f \in L(E', E)$, with E' complete. Then $f[E']$ is meagre in E or $f[E'] = E$, according to whether $f[G_{\vec{0}}(1)]$ is or is not nowhere dense.*[2]

Proof. If $f[G_0(1)]$ is nowhere dense in E, so also is $f[G_0(n)]$, $n > 0$. (Verify by Problems 15 and 19 in §7.) But then

$$f[E'] = f\Big[\bigcup_{n=1}^{\infty} G_0(n)\Big] = \bigcup_{n=1}^{\infty} f[G_{\vec{0}}(n)]$$

is a countable union of nowhere dense sets, hence meagre, by definition.

Now suppose $f[G_{\vec{0}}(1)]$ is not nowhere dense; so $\overline{f[G_{\vec{0}}(1)]}$ contains some $G_q(r) \subseteq E$. We may assume $q \in f[G_{\vec{0}}(1)]$ (if not, replace q by a close point from $f[G_{\vec{0}}(1)]$). Then $q = f(\vec{p})$ for some $\vec{p} \in G_{\vec{0}}(1)$. The latter implies

$$|-\vec{p}| = |\vec{p}| = \rho(\vec{p}, \vec{0}) < 1;$$

so

$$G_{-\vec{p}}(1) \subseteq G_{\vec{0}}(2).$$

Also, as $\overline{f[G_{\vec{0}}(1)]} \supseteq G_q(r)$, translation by $-q = f(-\vec{p})$ yields

$$\overline{f[G_{\vec{0}}(1)]} + f(-\vec{p}) \supseteq G_q(r) - q = G_0(r),$$

i.e.,

$$G_0(r) \subseteq \overline{f[G_{-\vec{p}}(1)]} \subseteq \overline{f[G_{\vec{0}}(2)]}.$$

Hence $\overline{f[G_{\vec{0}}(1)]} \supseteq G_0(\frac{1}{2}r)$ (why?); so, by the Lemma

$$f[G_{\vec{0}}(1)] \supseteq G_0\Big(\frac{1}{2}r\Big) \text{ in } E. \tag{1}$$

---

[2] Of course, if E is meagre, so is $f[E']$ in both cases.

This implies $f[G_{\vec{0}}(2n)] \supseteq G_0(nr)$, and so

$$f[E'] \supseteq \bigcup_{n=1}^{\infty} G_0(nr) = E,$$

i.e., $f[E'] = E$, as required. Thus the theorem is proved. $\square$

Theorem 3 (Open map principle)**.** *Let $f \in L(E', E)$, with E' and E complete. Then the map f is open on E' iff $f[E'] = E$, i.e., iff f is* onto *E.*

Proof. If $f[E'] = E$, then by Theorem 1, $f[E']$ is *nonmeagre* in E, as is E itself. Thus by Theorem 2, $f[G_{\vec{0}}(1)]$ is *not* nowhere dense, and (1) follows as before. Hence by Problems 15(iii) and 19 in §7, $f[G_{\vec{p}}] \supseteq$ some G_q whenever $q = f(\vec{p})$. (Why?) Therefore, $G_{\vec{p}} \subseteq A \subseteq E'$ implies

$$G_{f(\vec{p})} \subseteq f[G_{\vec{p}}] \subseteq f[A];$$

i.e., *f maps any interior point $\vec{p} \in A$ into such a point of $f[A]$*. By Problem 8 in §7, f is open on E'.

Conversely, if so, then $f[E']$ is an open set $\neq \emptyset$ in E, a *complete* space; so by Theorems 1 and 2, $f[E']$ is nonmeagre and equals E. (See also Problem 16(ii) in §7.) $\square$

Note 1. Theorem 3 holds even if f is not one-to-one.

Note 2. If in Theorem 3, however, f *is* bijective, it is open on E', and so $f^{-1} \in L(E, E')$ by Note 1 in §7. (*This is the promised* general *proof of Corollary* 2 *in* §6.)

Theorem 4 (Banach–Steinhaus uniform boundedness principle)**.** *Let E' be complete. Let $\mathcal{N}$ be a family of maps $f \in L(E', E)$ such that*

$$(\forall\, x \in E')\ (\exists\, k \in E^1)\ (\forall\, f \in \mathcal{N}) \quad |f(\vec{x})| < k. \tag{2}$$

(*"$\mathcal{N}$ is bounded at each $\vec{x}$."*)

Then $\mathcal{N}$ is "norm-bounded," i.e.,

$$(\exists\, K \in E^1)\ (\forall\, f \in \mathcal{N}) \quad \|f\| < K,$$

with $\|\ \|$ as in §2.

Proof. It suffices to show that $\mathcal{N}$ is "uniformly" bounded on some globe,

$$(\exists\, c \in E^1)\ (\exists\, G = G_{\vec{p}}(r))\ (\forall\, f \in \mathcal{N})\ (\forall\, \vec{x} \in G) \quad |f(\vec{x})| \leq c. \tag{3}$$

For then $|\vec{x} - \vec{p}| \leq r$ implies

$$2c > |f(\vec{x}) - f(\vec{p})| = |f(\vec{x} - \vec{p})|,$$

or (setting $\vec{x} - \vec{p} = r\vec{y}$) $|\vec{y}| < 1$ implies

$$(\forall\, f \in \mathcal{N}) \quad |f(\vec{y})| < \frac{2c}{r} \quad \text{(why?)};$$

so

$$(\forall\, f \in \mathcal{N}) \quad \|f\| = \sup_{|\vec{y}|\le 1} |f(\vec{y})| < \frac{2c}{r}.$$

Thus, seeking a contradiction, suppose (3) *fails* and assume its negation:

$$(4) \qquad (\forall\, c \in E^1)\ (\forall\, G = G_{\vec{p}}(r))\ (\exists\, f \in \mathcal{N})\ (\exists\, \vec{x} \in G = G_{\vec{p}}(r)) \quad |f(\vec{x})| > c.$$

Then for $c = 1$, we can fix some $f_1 \in \mathcal{N}$ and $G_{\vec{x}_1}(r_1)$ such that $0 < r_1 < 1$ and

$$|f_1(\vec{x}_1)| > 1.$$

By the continuity of the norm $|\ |$, we can choose r_1 so small that

$$(\forall\, \vec{x} \in \overline{G_{\vec{x}_1}(r_1)}) \quad |f(\vec{x})| > 1.$$

Again by (4), we fix $f_2 \in \mathcal{N}$ and $\vec{x}_2 \in G_{\vec{x}_1}(r_1)$ such that $|f_2| > 2$ on some globe

$$\overline{G_{\vec{x}_2}(r_2)} \subseteq G_{\vec{x}_1}(r_1),$$

with $0 < r_2 < 1/2$. Inductively, we thus form a *contracting* sequence of closed globes

$$\overline{G_{\vec{x}_n}(r_n)}, \quad 0 < r_n < \frac{1}{n},$$

and a sequence $\{f_n\} \subseteq \mathcal{N}$, such that

$$(\forall\, n) \quad |f_n| > n \text{ on } \overline{G_{\vec{x}_n}(r_n)} \subseteq E'.$$

As E' is complete, so are the closed globes $\overline{G_{\vec{x}_n}(r_n)} \subseteq E'$. Also, $0 < r_n < 1/n \to 0$. Thus by Cantor's theorem (Theorem 5 of Chapter 4, §6), there is

$$\vec{x}_0 \in \bigcap_{n=1}^{\infty} \overline{G_{\vec{x}_n}(r_n)}.$$

As $\vec{x}_0$ is in *each* $\overline{G_{\vec{x}_n}(r_n)}$, we have

$$(\forall\, n) \quad |f_n(\vec{x}_0)| > n;$$

so $\mathcal{N}$ *is not bounded at* $\vec{x}_0$, contrary to (2). This contradiction completes the proof. □

Note 3. Complete normed spaces are also called *Banach spaces*.

Problems on Baire Categories and Linear Maps

1. Verify the equivalence of the various formulations in Definition 1. Discuss: A is nowhere dense iff it is not dense in any open set $\neq \emptyset$.
2. Verify Examples (a) to (e). Show that Cantor's set P is *uncountable.* [Hint: Each $p \in P$ corresponds to a "ternary fraction," $p = \sum_{n=1}^{\infty} x_n/3^n$, also written $0.x_1, x_2, \ldots, x_n, \ldots$, where $x_n = 0$ or $x_n = 2$ according to whether p is to the left,

or to the right, of the nearest "removed" open interval of length $1/3^n$. Imitate the proof of Theorem 3 in Chapter 1, §9, for uncountability. See also Chapter 1, §9, Problem 2(ii).]

3. Complete the missing details in the proof of Theorems 1 to 4.

4. Prove the following.

(i) If $B \subseteq A$ and A is nowhere dense or meagre, so is B.

(ii) If $B \subseteq A$ and B is nonmeagre, so is A.
[Hint: Assume A is meagre and use (i)).]

(iii) Any finite union of nowhere dense sets is nowhere dense. Disprove it for infinite unions.

(iv) Any countable union of meagre sets is meagre.

5. Prove that in a *discrete* space (S, ρ), only $\emptyset$ is meagre.
[Hint: Use Problem 8 in Chapter 3, §17, Example 7 in Chapter 3, §12, and our present Theorem 1.]

6. Use Theorem 1 to give a new proof for the existence of irrationals in E^1.
[Hint: The rationals R are a meagre set, while E^1 is not.]

7. What is wrong about this "proof" that every *closed* set $F \neq \emptyset$ in a complete space (S, ρ) is residual: "By Theorem 5 of Chapter 3, §17, F is complete *as a subspace*. Thus by Theorem 1, F is residual." Give counterexamples!

8. We call K a *$\mathcal{G}_\delta$-set* and write $K \in \mathcal{G}_\delta$ iff $K = \bigcap_{n=1}^{\infty} G_n$ for some *open* sets G_n.[3]

(i) Prove that if K is a $\mathcal{G}_\delta$-set, and if K is dense in a complete metric space (S, ρ), i.e., $\overline{K} = S$, then K is residual in S.
[Hint: Let $F_n = -G_n$. Verify that $(\forall n)\ G_n$ is dense in S, and F_n is nowhere dense. Deduce that $-K = -\bigcap G_n = \bigcup F_n$ is meagre. Use Theorem 1.]

(ii) Infer that R (the rationals) is *not* a $\mathcal{G}_\delta$-set in E^1 (cf. Example (c)).

9. Show that, in a complete metric space (S, ρ), a meagre set A cannot have interior points.
[Hint: Otherwise, A would obtain a *globe* G. Use Theorem 1 and Problem 4(ii).]

10. (i) A *singleton* $\{p\} \subseteq (S, \rho)$ is nowhere dense if S clusters at p; otherwise, it is *nonmeagre* in S (being a *globe*, and not a union of nowhere dense sets).

(ii) If $A \subseteq S$ clusters at *each* $p \in A$, any countable set $B \subseteq A$ is meagre in S.

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[3] Such is any *closed* set $A = \overline{A} \subseteq (S, \rho)$ (see Problem 20 in Chapter 3, §16).

11. (i) Show that if $\emptyset \neq A \in \mathcal{G}_\delta$ (see Problem 8) in a *complete* space (S, ρ), and A clusters at each $p \in A$, then A is uncountable.

(ii) Prove that any nonempty *perfect* set (Chapter 3, §14) in a complete space is uncountable.

(iii) How about R (the rationals) in E^1 and in R as a subspace of E^1? What is wrong?

[Hints: (i) The subspace $(\overline{A}, \rho)$ is complete (why?); so A is nonmeagre in $\overline{A}$, by Problem 8. Use Problem 10(ii). (ii) Use Footnote 3.]

12. If G is open in (S, ρ), then $\overline{G} - G$ is nowhere dense in S.
[Hint: $\overline{G} - G = \overline{G} \cap (-G)$ is *closed*; so

$$\overline{(\overline{G} - G)^0} = (\overline{G} - G)^0 = (\overline{G} \cap -G)^0 = \emptyset$$

by Problem 15 in Chapter 3, §12 and Problem 15 in Chapter 3, §16.]

13. ("Simplified" uniform boundedness theorem.) Let $f_n \colon (S, \rho) \to (T, \rho')$ be continuous for $n = 1, 2, \ldots$, with S complete. If $\{f_n(x)\}$ is a bounded sequence in T for each $x \in S$, then $\{f_n\}$ is *uniformly* bounded on some open $G \neq \emptyset$:

$$(\forall\, p \in T)\ (\exists\, k)\ (\forall\, n)\ (\forall\, x \in G) \quad \rho'(p, f_n(x)) \leq k.$$

[Outline: Fix $p \in T$ and $(\forall\, n)$ set

$$F_n = \{x \in S \mid (\forall\, m)\ n \geq \rho'(p, f_m(x))\}.$$

Use the continuity of f_m and of ρ' to show that F_n is closed in S, and $S = \bigcup_{n=1}^{\infty} F_n$. By Theorem 1, S is nonmeagre; so at least one F_n is *not* nowhere dense—call it F, so $(\overline{F})^0 = F^0 \neq \emptyset$. Set $G = F^0$ and show that G is as required.]

14. Let $f_n \colon (S, \rho) \to (T, \rho')$ be continuous for $n = 1, 2, \ldots$ Show that if $f_n \to f$ (pointwise) on S, then f is continuous on $S - Q$, with Q meagre in S.
[Outline: $(\forall\, k, m)$ let

$$A_{km} = \bigcup_{m=n}^{\infty} \left\{ x \in S \,\middle|\, \rho'(f_n(x), f_m(x)) > \frac{1}{k} \right\}.$$

By the continuity of ρ', f_n and f_m, A_{km} is open in S. (Why?) So by Problem 12, $\bigcup_{m=1}^{\infty} (\overline{A_{km}} - A_{km})$ is meagre for $k = 1, 2, \ldots$.

Also, as $f_n \to f$ on S, $\bigcap_{m=1}^{\infty} A_{km} = \emptyset$. (Verify!) Thus

$$(\forall\, k) \quad \bigcap_{m=1}^{\infty} \overline{A_{km}} \subseteq \bigcup_{m=1}^{\infty} (\overline{A_{km}} - A_{km}).$$

(Why?) Hence the set $Q = \bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \overline{A_{km}}$ is meagre in S.

Moreover, $S - Q = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} (-A_{km})^0$ by Problem 16 in Chapter 3, §16. Deduce that if $p \in S - Q$, then

$$(\forall\, \varepsilon > 0)\ (\exists\, m_0)\ (\exists\, G_p)\ (\forall\, n, m \geq m_0)\ (\forall\, x \in G_0) \quad \rho'(f_m(x), f_n(x)) < \varepsilon.$$

Keeping m fixed, let $n \to \infty$ to get

$$(\forall \varepsilon > 0)\ (\exists m_0)\ (\exists G_p)\ (\forall m \geq m_0)\ (\forall x \in G_p) \quad \rho'(f_m(x), f(x)) \leq \varepsilon.$$

Now modify the proof of Theorem 2 of Chapter 4, §12, to show that this implies the continuity of f at each $p \in S - Q$.]

§9. Local Extrema. Maxima and Minima

We say that $f\colon E' \to E^1$ has a *local maximum* (*minimum*) at $\vec{p} \in E'$ iff $f(\vec{p})$ is the largest (least) value of f *on some globe G about $\vec{p}$*; more precisely, iff

$$(\forall \vec{x} \in G) \quad \Delta f = f(\vec{x}) - f(\vec{p}) < 0\ (> 0).$$

We speak of an *improper extremum* if we only have $\Delta f \leq 0$ (≥ 0) on G. In any case, all depends on the *sign* of Δf.

From Problem 6 in §1, recall the following *necessary* condition.

Theorem 1. *If $f\colon E' \to E^1$ has a local extremum at $\vec{p}$ then $D_{\vec{u}} f(\vec{p}) = 0$ for all $\vec{u} \neq \vec{0}$ in E'.*

In the case $E' = E^n$ (C^n), this means that $d^1 f(\vec{p}, \cdot\,) = 0$ on E'.

(Recall that $d^1 f(\vec{p}; \vec{t}) = \sum_{k=1}^{n} D_k f(\vec{p}) t_k$. It vanishes if the $D_k f(\vec{p})$ do.)

Note 1. This condition is only necessary, not sufficient. For example, if $f(x, y) = xy$, then $d^1 f(\vec{0}, \cdot\,) = 0$; yet f has no extremum at $\vec{0}$. (Verify!)

Sufficient conditions were given in Theorem 2 of §5, for $E' = E^1$. We now take up $E' = E^2$.

Theorem 2. *Let $f\colon E^2 \to E^1$ be of class CD^2 on a globe $G = G_{\vec{p}}(\delta)$. Suppose $d^1 f(\vec{p}, \cdot\,) = 0$ on E^2. Set $A = D_{11} f(\vec{p})$, $B = D_{12} f(\vec{p})$, and $C = D_{22} f(\vec{p})$.*

Then the following statements are true.

(i) *If $AC > B^2$, f has a maximum or minimum at $\vec{p}$, according to whether $A < 0$ or $A > 0$.*

(ii) *If $AC < B^2$, f has no extremum at $\vec{p}$.*

The case $AC = B$ is unresolved.

Proof. Let $\vec{x} \in G$ and $\vec{u} = \vec{x} - \vec{p} \neq \vec{0}$.

As $d^1 f(\vec{p}, \cdot\,) = 0$, Theorem 2 in §5, yields

$$\Delta f = f(\vec{x}) - f(\vec{p}) = R_1 = \frac{1}{2} d^2 f(\vec{s}; \vec{u}),$$

with $\vec{s} \in L(\vec{p}, \vec{x}) \subseteq G$ (see Corollary 1 of §5). As $f \in CD^2$, we have $D_{12} f = D_{21} f$ on G (Theorem 1 in §5). Thus by formula (4) in §5,

$$\Delta f = \frac{1}{2} d^2 f(\vec{s}; \vec{u}) = \frac{1}{2}[D_{11} f(\vec{s}) u_1^2 + 2 D_{12} f(\vec{s}) u_1 u_2 + D_{22} f(\vec{s}) u_2^2]. \tag{1}$$

Now, as the partials involved are continuous, we can choose $G = G_{\vec{p}}(\delta)$ so small that the sign of expression (1) will not change if $\vec{s}$ is replaced by $\vec{p}$. Then the crucial sign of Δf on G coincides with that of

$$D = Au_1^2 + 2Bu_1u_2 + Cu_2^2 \tag{2}$$

(with A, B, and C as stated in the theorem).

From (2) we obtain, by elementary algebra,

$$AD = (Au_1 + Bu_2)^2 + (AC - B^2)u_2^2, \tag{3}$$

$$CD = (Cu_1 + Bu_2)^2 + (AC - B^2)u_2^2. \tag{3'}$$

Clearly, if $AC > B^2$, the right-side expression in (3) is > 0; so $AD > 0$, i.e., *D has the same sign as A.*

Hence if $A < 0$, we also have $\Delta f < 0$ on G, and f has a maximum at $\vec{p}$. If $A > 0$, then $\Delta f > 0$, and f has a minimum at $\vec{p}$.

Now let $AC < B^2$. We claim that no matter how small $G = G_{\vec{p}}(\delta)$, Δf *changes* sign as $\vec{x}$ varies in G, and so f has no extremum at $\vec{p}$.

Indeed, we have $\vec{x} = \vec{p} + \vec{u}$, $\vec{u} = (u_1, u_2) \neq \vec{0}$. If $u_2 = 0$, (3) shows that D and Δf have *the same* sign as A $(A \neq 0)$.

But if $u_2 \neq 0$ and $u_1 = -Bu_2/A$ (assuming $A \neq 0$), then D and Δf have the sign *opposite* to that of A; and $\vec{x}$ is still in G if u_2 is small enough (how small?).

One proceeds similarly if $C \neq 0$ (interchange A and C, and use (3′).

Finally, if $A = C = 0$, then by (2), $D = 2Bu_1u_2$ and $B \neq 0$ (since $AC < B^2$). Again D and Δf *change* sign as u_1u_2 does; so f has no extremum at $\vec{p}$. Thus all is proved. $\square$

Briefly, the proof utilizes the fact that the *trinomial* (2) *is sign-changing iff its discriminant $B^2 - AC$ is positive*, i.e., $\begin{vmatrix} A & B \\ B & C \end{vmatrix} < 0$.

Note 2. Functions $f\colon C \to E^1$ (of one *complex* variable) are likewise covered by Theorem 2 if one treats them as functions *on* E^2 (of two real variables).

Functions of *n* variables. Here we must rely on the algebraic theory of so-called *symmetric quadratic forms*, i.e., polynomials $P\colon E^n \to E^1$ of the form

$$P(\vec{u}) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} u_i u_j,$$

where $\vec{u} = (u_i, \ldots, u_n) \in E^n$ and $a_{ij} = a_{ji} \in E^1$.

We take for granted a theorem due to J. J. Sylvester (see S. Perlis, *Theory of Matrices*, 1952, p. 197), which may be stated as follows.

Let $P\colon E^n \to E^1$ *be a symmetric quadratic form,*

$$P(\vec{u}) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} u_i u_j.$$

(i) $P > 0$ *on all of* $E^n - \{\vec{0}\}$ *iff the following* n *determinants* A_k *are positive:*

$$A_k = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{vmatrix}, \quad k = 1, 2, \dots, n. \tag{4}$$

(ii) *We have* $P < 0$ *on* $E^n - \{\vec{0}\}$ *iff* $(-1)^k A_k > 0$ *for* $k = 1, 2, \dots, n$.

Now we can extend Theorem 2 to the case $f\colon E^n \to E^1$. (This will also cover $f\colon C^n \to E^1$, treated as $f\colon E^{2n} \to E^1$.) The proof resembles that of Theorem 2.

Theorem 3. *Let* $f\colon E^n \to E^1$ *be of class* CD^2 *on some* $G = G_{\vec{p}}(\delta)$. *Suppose* $df(\vec{p}, \cdot\,) = 0$ *on* E^n. *Define the* A_k *as in* (4), *with* $a_{ij} = D_{ij} f(\vec{p})$, $i, j, k \le n$. *Then the following statements hold.*

(i) f *has a local minimum at* $\vec{p}$ *if* $A_k > 0$ *for* $k = 1, 2, \dots, n$.

(ii) f *has a local maximum at* $\vec{p}$ *if* $(-1)^k A_k > 0$ *for* $k = 1, \dots, n$.

(iii) f *has no extremum at* $\vec{p}$ *if the expression*

$$P(\vec{u}) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} u_i u_j$$

is > 0 *for* some $\vec{u} \in E^n$ *and* < 0 *for* others (*i.e.*, P changes sign *on* E^n).

Proof. Let again $\vec{x} \in G$, $\vec{u} = \vec{x} - \vec{p} \neq \vec{0}$, and use Taylor's theorem to obtain

$$\Delta f = f(\vec{x}) - f(\vec{p}) = R_1 = \frac{1}{2} d^2 f(\vec{s}; \vec{u}) = \sum_{j=1}^{n} \sum_{i=1}^{n} D_{ij} f(\vec{s}) u_i u_j, \tag{5}$$

with $\vec{s} \in L(\vec{x}, \vec{p})$.

As $f \in CD^2$, the partials $D_{ij} f$ are continuous on G. Thus we can make G so small that the sign of the last double sum does not change if $\vec{s}$ is replaced by $\vec{p}$. Hence the sign of Δf on G is the same as that of $P(\vec{u}) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} u_i u_j$, with the a_{ij} as stated in the theorem.

The quadratic form P is symmetric since $a_{ij} = a_{ji}$ by Theorem 1 in §5. Thus by Sylvester's theorem stated above, one easily obtains our assertions (i) and (ii). Indeed, they are immediate from clauses (i) and (ii) of that theorem.

Now, for (iii), suppose $P(\vec{u}) > 0 > P(\vec{v})$, i.e.,

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}u_iu_j > 0 > \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}v_iv_j \quad \text{for some } \vec{u}, \vec{v} \in E^n - \{\vec{0}\}.$$

If here $\vec{u}$ and $\vec{v}$ are replaced by $t\vec{u}$ and $t\vec{v}$ $(t \neq 0)$, then u_iu_j and v_iv_j turn into $t^2u_iu_j$ and $t^2v_iv_j$, respectively. Hence

$$P(t\vec{u}) = t^2P(\vec{u}) > 0 > t^2P(\vec{v}) = P(t\vec{v}).$$

Now, for any $t \in (0, \delta/|\vec{u}|)$, the point $\vec{x} = \vec{p} + t\vec{u}$ lies on the $\vec{u}$-directed line through $\vec{p}$, *inside* $G = G_{\vec{p}}(\delta)$. (Why?) Similarly for the point $\vec{x}\,' = \vec{p} + t\vec{v}$.

Hence for such $\vec{x}$ and $\vec{x}\,'$, Taylor's theorem again yields formulas analogous to (5) for some $\vec{s} \in L(\vec{p}, \vec{x})$ and $\vec{s}\,' \in L(\vec{p}, \vec{x}\,')$ lying *on the same two lines*. It again follows that for small δ,

$$f(\vec{x}) - f(\vec{p}) > 0 > f(\vec{x}\,') - f(\vec{p}),$$

just as $P(\vec{u}) > 0 > P(\vec{v})$.

Thus Δf *changes* sign on $G_{\vec{p}}(\delta)$, and (iii) is proved. $\square$

Note 3. Still unresolved are cases in which $P(\vec{u})$ *vanishes* for some $\vec{u} \neq \vec{0}$, *without* changing its sign; e.g., $P(\vec{u}) = (u_1 + u_2 + u_3)^2 = 0$ for $\vec{u} = (1, 1, -2)$. Then the answer depends on *higher-order* terms of the Taylor formula. In particular, if $d^1f(\vec{p}, \cdot\,) = d^2f(\vec{p}, \cdot\,) = 0$ on E^n, then $\Delta f = R_2 = \frac{1}{6}d^3f(\vec{p}; \vec{s})$, etc.

Note 4. The *largest* or *least* value of f on a set A (sometimes called the *absolute* maximum or minimum) may occur at some *noninterior* (e.g., boundary) point $\vec{p} \in A$, and then fails to be among the *local* extrema (where, by definition, a *globe* $G_{\vec{p}} \subseteq A$ is presupposed). Thus to find *absolute* extrema, one must also explore the behaviour of f at *noninterior* points of A.

By Theorem 1, local extrema can occur only at so-called *critical* points $\vec{p}$, i.e., those at which all directional derivatives vanish (*or fail to exist*, in which case $D_{\vec{u}}f(\vec{p}) = 0$ *by convention*).

In practice, to find such points in E^n (C^n), one equates the *partials* D_kf $(k \leq n)$ to 0. Then one uses Theorems 2 and 3 *or other considerations* to determine whether an extremum really exists.

Examples.

(A) Find the largest value of

$$f(x, y) = \sin x + \sin y - \sin(x + y)$$

on the set $A \subseteq E^2$ bounded by the lines $x = 0$, $y = 0$ and $x + y = 2\pi$.

We have

$$D_1f(x, y) = \cos x - \cos(x + y) \text{ and } D_2f(x, y) = \cos y - \cos(x + y).$$

Inside the triangle A, both partials vanish *only* at the point $(\frac{2\pi}{3}, \frac{2\pi}{3})$ at which $f = \frac{3}{2}\sqrt{3}$. *On the boundary* of A (i.e., on the lines $x = 0$, $y = 0$ and $x+y = 2\pi$), $f = 0$. Thus even without using Theorem 2, it is evident that f attains its *largest* value,

$$f\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) = \frac{3}{2}\sqrt{3},$$

at this *unique* critical point.

(B) Find the largest and the least value of

$$f(x, y, z) = a^2x^2 + b^2y^2 + c^2z^2 - (ax^2 + by^2 + cz^2)^2,$$

on the condition that $x^2 + y^2 + z^2 = 1$ and $a > b > c > 0$.

As $z^2 = 1 - x^2 - y^2$, we can eliminate z from $f(x, y, z)$ and replace f by $F\colon E^2 \to E^1$:

$$F(x, y) = (a^2 - c^2)x^2 + (b^2 - c^2)y^2 + c^2 - [(a - c)x^2 + (b - c)y^2 + c]^2.$$

(Explain!) For F, we seek the extrema on the disc $\overline{G} = \overline{G}_0(1) \subset E^2$, *where* $x^2 + y^2 \le 1$ (so as not to violate the condition $x^2 + y^2 + z^2 = 1$).

Equating to 0 the two partials

$$D_1F(x, y) = 2x(a - c)\{(a + c) - 2[(a - c)x^2 + (b - c)y^2 + c]^2\} = 0,$$
$$D_2F(x, y) = 2y(b - c)\{(b + c) - 2[(a - c)x^2 + (b - c)y^2 + c]^2\} = 0$$

and solving this system of equations, we find these critical points *inside* G:

(1) $x = y = 0$ $(F = 0)$;

(2) $x = 0$, $y = \pm 2^{-\frac{1}{2}}$ $(F = \frac{1}{4}(b - c)^2)$; and

(3) $x = \pm 2^{-\frac{1}{2}}$, $y = 0$ $(F = \frac{1}{4}(a - c)^2)$.

(Verify!)

Now, for the *boundary* of $\overline{G}$, i.e., the circle $x^2 + y^2 = 1$, *repeat* this process: substitute $y^2 = 1 - x^2$ in the formula for $F(x, y)$, thus reducing it to

$$h(x) = (a^2 - b^2)x^2 + b^2 + [(a - b)x^2 + b]^2, \quad h\colon E^1 \to E^1,$$

on the interval $[-1, 1] \subset E^1$. *In* $(-1, 1)$ the derivative

$$h'(x) = 2(a - b)x(1 - 2x^2)$$

vanishes only when

(4) $x = 0$ $(h = 0)$, and

(5) $x = \pm 2^{-\frac{1}{2}}$ $(h = \frac{1}{4}(a - b)^2)$.

Finally, at the *endpoints* of $[-1, 1]$, we have

(6) $x = \pm 1$ $(h = 0)$.

Comparing the resulting function values in all six cases, we conclude that the least of them is 0, while the largest is $\frac{1}{4}(a-c)^2$. These are the desired least and largest values of f, subject to the conditions stated. They are attained, respectively, at the points

$$(0, 0, \pm 1),\ (0, \pm 1, 0),\ (\pm 1, 0, 0),\ \text{and}\ (\pm 2^{-\frac{1}{2}}, 0, \pm 2^{-\frac{1}{2}}).$$

Again, the use of Theorems 2 and 3 was redundant.[1] However, we suggest as an exercise that the reader test the critical points *of F* by using Theorem 2.

Caution. Theorems 1 to 3 apply to functions of *independent* variables only. In Example (B), x, y, z were made *interdependent* by the imposed equation

$$x^2 + y^2 + z^2 = 1$$

(which *geometrically* limits all to the surface of $G_{\vec{0}}(1)$ in E^3), so that one of them, z, could be *eliminated*. Only *then* can Theorems 1 to 3 be used.

Problems on Maxima and Minima

1. Verify Note 1.

1′. Complete the missing details in the proof of Theorems 2 and 3.

2. Verify Examples (A) and (B). Supplement Example (A) by applying Theorem 2.

3. Test f for extrema in E^2 if $f(x, y)$ is

(i) $\dfrac{x^2}{2p} + \dfrac{y^2}{2q}$ $(p > 0,\ q > 0)$;

(ii) $\dfrac{x^2}{2p} - \dfrac{y^2}{2q}$ $(p > 0,\ q > 0)$;

(iii) $y^2 + x^4$;

(iv) $y^2 + x^3$.

4. (i) Find the maximum volume of an interval $A \subset E^3$ (see Chapter 3, §7) whose edge lengths x, y, z have a prescribed sum: $x+y+z = a$.

(ii) Do the same in E^4 and in E^n; show that A is a *cube*.

---

[1] Indeed, by Theorem 2(ii) in Chapter 4, §8, absolute extrema *must* exist here, as all is limited to the *compact* sphere, $x^2 + y^2 + z^2 = 1$.

(iii) Hence deduce that

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{1}{n} \sum_1^n x_k \quad (x_k \geq 0),$$

i.e., *the geometric mean of* n *nonnegative numbers is* $\leq$ *their arithmetic mean.*

5. Find the minimum value for the sum $f(x, y, z, t) = x + y + z + t$ of four positive numbers on the condition that $xyzt = c^4$ (constant).
[Answer: $x = y = z = t = c$; $f_{\max} = 4c$.]

6. Among all triangles inscribed in a circle of radius R, find the one of maximum area.
[Hint: Connect the vertices with the center. Let x, y, z be the angles at the center. Show that the area of the triangle $= \frac{1}{2}R^2(\sin x + \sin y + \sin z)$, with $z = 2\pi - (x + y)$.]

7. Among all intervals $A \subset E^3$ inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

find the one of largest volume.
[Answer: the edge lengths are $\frac{2a}{\sqrt{3}}$, $\frac{2b}{\sqrt{3}}$, $\frac{2c}{\sqrt{3}}$.]

8. Let $P_i = (a_i.b_i)$, $i = 1, 2, 3$, be 3 points in E^2 forming a triangle in which one angle (say, $\measuredangle P_1$) is $\geq 2\pi/3$.

Find a point $P = (x, y)$ for which the sum of the distances,

$$PP_1 + PP_2 + PP_3 = \sum_{i=1}^3 \sqrt{(x - a_i)^2 + (y - b_i)^2},$$

is the least possible.
[Outline: Let $f(x, y) = \sum_{i=1}^3 \sqrt{(x - a_i)^2 + (y - b_i)^2}$.

Show that f *has no partial derivatives* at P_1, P_2, or P_3 (and so P_1, P_2, and P_3 are *critical points* at which an extremum *may* occur), while at other points P, partials do exist but never vanish *simultaneously*, so that there are no other *critical* points.

Indeed, prove that $D_1 f(P) = 0 = D_2 f(P)$ would imply that

$$\sum_{i=1}^3 \cos\theta_i = 0 = \sum_1^3 \sin\theta_i,$$

where θ_i is the angle between $\overline{PP_i}$ and the x-axis; hence

$$\sin(\theta_1 - \theta_2) = \sin(\theta_2 - \theta_3) = \sin(\theta_3 - \theta_1) \quad \text{(why?)},$$

and so $\theta_1 - \theta_2 = \theta_2 - \theta_3 = \theta_3 - \theta_1 = 2\pi/3$, contrary to $\measuredangle P_1 \geq 2\pi/3$. (Why?)

From geometric considerations, conclude that f has an absolute minimum at P_1. (This shows that *one cannot disregard points at which* f *has no partials*.)]

9. Continuing Problem 8, show that if none of $\measuredangle P_1$, $\measuredangle P_2$, and $\measuredangle P_3$ is $\geq 2\pi/3$, then f attains its least value at some P (*inside* the triangle) such that $\measuredangle P_1PP_2 = \measuredangle P_2PP_3 = \measuredangle P_3PP_1 = 2\pi/3$.

[Hint: Verify that $D_1 f = 0 = D_2 f$ at P.

Use the law of cosines to show that $P_1P_2 > PP_2 + \frac{1}{2}PP_1$ and $P_1P_3 > PP_3 + \frac{1}{2}PP_1$.

Adding, obtain $P_1P_3 + P_1P_2 > PP_1 + PP_2 + PP_3$, i.e., $f(P_1) > f(P)$. Similarly, $f(P_2) > f(P)$ and $f(P_3) > f(P)$.

Combining with Problem 8, obtain the result.]

10. In a circle of radius R inscribe a polygon with $n + 1$ sides of maximum area.

[Outline: Let $x_1, x_2, \ldots, x_{n+1}$ be the central angles subtended by the sides of the polygon. Then its area A is

$$\frac{1}{2}R^2 \sum_{k=1}^{n+1} \sin x_k,$$

with $x_{n+1} = 2\pi - \sum_{k=1}^{n} x_k$. (Why?) Thus all reduces to maximizing

$$f(x_1, \ldots, x_n) = \sum_{k=1}^{n} \sin x_k + \sin\Big(2\pi - \sum_{k=1}^{n} x_k\Big),$$

on the condition that $0 \leq x_k$ and $\sum_{k=1}^{n} x_k \leq 2\pi$. (Why?)

These inequalities define a bounded set $D \subset E^n$ (called a *simplex*). Equating all partials of f to 0, show that the only critical point *interior* to D is $\vec{x} = (x_1, \ldots, x_n)$, with $x_k = \frac{2\pi}{n+1}$, $k \leq n$ (implying that $x_{n+1} = \frac{2\pi}{n+1}$, too). For *that* $\vec{x}$, we get

$$f(\vec{x}) = (n+1)\sin[2\pi/(n+1)].$$

This value must be compared with the "boundary" values of f, on the "faces" of the simplex D (see Note 4).

Do this by induction. For $n = 2$, Problem 6 shows that $f(\vec{x})$ is indeed the *largest* when all x_k equal $\frac{2\pi}{n+1}$. Now let D_n be the "face" of D, where $x_n = 0$. On that face, treat f as a function of only $n - 1$ variables, $x_1, \ldots, x_{n-1}$.

By the inductive hypothesis, the largest value of f on D_n is $n\sin(2\pi/n)$. Similarly for the other "faces." As $n\sin(2\pi/n) < (n+1)\sin 2\pi/(n+1)$, the induction is complete.

Thus, the area A is the largest when the polygon is *regular*, for which

$$A = \frac{1}{2}R^2(n+1)\sin\frac{2\pi}{n+1}.]$$

11. Among all triangles of a prescribed perimeter $2p$, find the one of maximum area.

[Hint: Maximize $p(p-x)(p-y)(p-z)$ on the condition that $x + y + z = 2p$.]

12. Among all triangles of area A, find the one of smallest perimeter.

13. Find the shortest distance from a given point $\vec{p} \in E^n$ to a given plane $\vec{u} \cdot \vec{x} = c$ (Chapter 3, §§4–6). Answer:

$$\pm\frac{\vec{u} \cdot \vec{p} - c}{|\vec{u}|}.$$

[Hint: First do it in E^3, writing (x, y, z) for $\vec{x}$.]

§10. More on Implicit Differentiation. Conditional Extrema

I. Implicit differentiation was sketched in §7. Under suitable assumptions (Theorem 4 in §7), one can differentiate a given system of equations,

$$g_k(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0, \quad k = 1, 2, \ldots, n, \tag{1}$$

treating the x_j as *implicit* functions of the y_i without seeking an *explicit* solution of the form

$$x_j = H_j(y_1, \ldots, y_m).$$

This yields a new system of equations from which the partials $D_i H_j = \dfrac{\partial x_j}{\partial y_i}$ can be found directly.

We now supplement Theorem 4 in §7 (review it!) by showing that this new system is *linear* in the partials involved and that its determinant is $\neq 0$. Thus in general, it is simpler to solve than (1).

As in Part IV of §7, we set

$$(\vec{x}, \vec{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \text{ and } g = (g_1, \ldots, g_n),$$

replacing the f of §7 by g. Then equations (1) simplify to

$$g(\vec{x}, \vec{y}) = \vec{0}, \tag{2}$$

where $g\colon E^{n+m} \to E^n$ (or $g\colon C^{n+m} \to C^n$).

Theorem 1 (implicit differentiation)**.** *Adopt all assumptions of Theorem 4 in* §7, *replacing f by g and setting $H = (H_1, \ldots, H_n)$,*

$$D_j g_k(\vec{p}, \vec{q}) = a_{jk}, \quad j \leq n + m, \ k \leq n.$$

Then for each $i = 1, \ldots, m$, we have n linear equations,

$$\sum_{j=1}^{n} a_{jk} D_i H_j(\vec{q}) = -a_{n+i,k}, \quad k \leq n, \tag{3}$$

with

$$\det(a_{jk}) \neq 0, \quad (j, k \leq n),$$

that uniquely determine the partials $D_i H_j(\vec{q})$ for $j = 1, 2, \ldots, n$.

Proof. As usual, extend the map $H\colon Q \to P$ of Theorem 4 in §7 to $H\colon E^m \to E^n$ (or $C^m \to C^n$) by setting $H = \vec{0}$ on $-Q$.

Also, define $\sigma\colon E^m \to E^{n+m}$ $(C^m \to C^{n+m})$ by

$$\sigma(\vec{y}) = (H(\vec{y}), \vec{y}) = (H_1(\vec{y}), \ldots, H_n(\vec{y}), y_1, \ldots, y_m), \quad \vec{y} \in E^m(C^m). \tag{4}$$

Then σ is differentiable at $\vec{q} \in Q$, as are its $n+m$ components. (Why?) Since $\vec{x} = H(\vec{y})$ is a solution of (2), equations (1) and (2) become *identities* when $\vec{x}$ is replaced by $H(\vec{y})$. Also, $\sigma(\vec{q}) = (H(\vec{q}), \vec{q}) = (\vec{p}, \vec{q})$ since $H(\vec{q}) = \vec{p}$. Moreover,

$$g(\sigma(\vec{y})) = g(H(\vec{y}), \vec{y}) = \vec{0} \text{ for } \vec{y} \in Q;$$

i.e., $g \circ \sigma = \vec{0}$ on Q.

Now, by assumption, $g \in CD^1$ at $(\vec{p}, \vec{q})$; so the chain rule (Theorem 2 in §4) applies, with f, $\vec{p}$, $\vec{q}$, n, and m replaced by σ, $\vec{q}$, $(\vec{p}, \vec{q})$, m, and $n+m$, respectively.

As $h = g \circ \sigma = \vec{0}$ on Q, an *open* set, the partials of h vanish on Q. So by Theorem 2 of §4, writing σ_j for the jth component of σ,

$$\vec{0} = \sum_{j=1}^{n+m} D_j g(\vec{p}, \vec{q}) \cdot D_i \sigma_j(\vec{q}), \quad i \leq m. \tag{5}$$

By (4), $\sigma_j = H_j$ if $j \leq n$, and $\sigma_j(\vec{y}) = y_i$ if $j = n+i$. Thus $D_i \sigma_j = D_i H_j$, $j \leq n$; but for $j > n$, we have $D_i \sigma_j = 1$ if $j = n+i$, and $D_i \sigma_j = 0$ otherwise. Hence by (5),

$$\vec{0} = \sum_{j=1}^{n} D_j g(\vec{p}, \vec{q}) \cdot D_i H_j(\vec{q}) + D_{n+i} g(\vec{p}, \vec{q}), \quad i = 1, 2, \ldots, m.$$

As $g = (g_1, \ldots, g_n)$, each of these vector equations splits into n scalar ones:

$$0 = \sum_{j=1}^{n} D_j g_k(\vec{p}, \vec{q}) \cdot D_i H_j(\vec{q}) + D_{n+i} g_k(\vec{p}, \vec{q}), \quad i \leq m, \ k \leq n. \tag{6}$$

With $D_j g_k(\vec{p}, \vec{q}) = a_{jk}$, this yields (3), where $\det(a_{jk}) = \det(D_j g_k(\vec{p}, \vec{q})) \neq 0$, by hypothesis (see Theorem 4 in §7).

Thus all is proved. $\square$

Note 1. By continuity (Note 1 in §6), we have $\det(D_j g_k(\vec{x}, \vec{y})) \neq 0$ for all $(\vec{x}, \vec{y})$ in a sufficiently small neighborhood of $(\vec{p}, \vec{q})$. Thus Theorem 1 *holds also with* $(\vec{p}, \vec{q})$ *replaced by such* $(\vec{x}, \vec{y})$. In practice, one does not have to memorize (3), but one obtains it by implicitly differentiating equations (1).

II. We shall now apply Theorem 1 to the theory of *conditional extrema.*

Definition 1.

We say that $f\colon E^{n+m} \to E^1$ has a local *conditional* maximum (minimum) at $\vec{p} \in E^{n+m}$, with *constraints*

$$g = (g_1, \ldots, g_n) = \vec{0}$$

($g : E^{n+m} \to E^n$) iff in some neighborhood G of $\vec{p}$ we have

$$\Delta f = f(\vec{x}) - f(\vec{p}) \leq 0 \quad (\geq 0, \text{ respectively})$$

for all $\vec{x} \in G$ *for which* $g(\vec{x}) = \vec{0}$.

In §9 (Example (B) and Problems), we found such conditional extrema by using the constraint equations $g = \vec{0}$ to eliminate some variables and thus reduce all to finding the *unconditional* extrema of a function of *fewer* (independent) variables.

Often, however, such elimination is cumbersome since it involves solving a system (1) of possibly nonlinear equations. It is here that implicit differentiation (based on Theorem 1) is useful.

Lagrange invented a method (known as that of *multipliers*) for finding the *critical points* at which such extrema *may* exist; to wit, we have the following: Given $f \colon E^{n+m} \to E^1$, set

$$F = f + \sum_{k=1}^{n} c_k g_k, \tag{7}$$

where the constants c_k *are to be determined* and g_k are as above.

Then find the partials $D_j F$ ($j \leq n + m$) and solve the system of $2n + m$ equations

$$D_j F(\vec{x}) = 0, \quad j \leq n + m, \quad \text{and} \quad g_k(\vec{x}) = 0, \quad k \leq n, \tag{8}$$

for the $2n + m$ "unknowns" x_j ($j \leq n + m$) and c_k ($k \leq n$), the c_k originating from (7).

Any $\vec{x}$ satisfying (8), with the c_k so determined is a critical point (*still to be tested*). The method is based on Theorem 2 below, where we again write $(\vec{p}, \vec{q})$ for $\vec{p}$ and $(\vec{x}, \vec{y})$ for $\vec{x}$ (we call it "*double notation*").

Theorem 2 (Lagrange multipliers)**.** *Suppose* $f \colon E^{n+m} \to E^1$ *is differentiable at*

$$(\vec{p}, \vec{q}) = (p_1, \ldots, p_n, q_1, \ldots, q_m)$$

and has a local extremum at $(\vec{p}, \vec{q})$ *subject to the constraints*

$$g = (g_1, \ldots, g_n) = \vec{0},$$

with g *as in Theorem* 1, $g \colon E^{n+m} \to E^n$. *Then*

$$\sum_{k=1}^{n} c_k D_j g_k(\vec{p}, \vec{q}) = -D_j f(\vec{p}, \vec{q}), \quad j = 1, 2, \ldots, n + m,^{1} \tag{9}$$

for certain multipliers c_k (*determined by the first* n *equations in* (9)).

---

[1] That is, $D_j F(\vec{p}, \vec{q}) = 0$, with F as in (7).

Proof. These n equations admit a unique solution for the c_k, as they are *linear*, and

$$\det(D_j g_k(\vec{p},\vec{q})) \neq 0 \quad (j, k \leq n)$$

by hypothesis. With the c_k so determined, (9) holds for $j \leq n$. It remains to prove (9) for $n < j \leq n+m$.

Now, since f has a conditional extremum at $(\vec{p},\vec{q})$ as stated, we have

$$f(\vec{x},\vec{y}) - f(\vec{p},\vec{q}) \leq 0 \ \ (\text{or } \ \geq 0) \tag{10}$$

for all $(\vec{x},\vec{y}) \in P \times Q$ with $g(\vec{x},\vec{y}) = \vec{0}$, provided we make the neighborhood $P \times Q$ small enough.

Define H and σ as in the previous proof (see (4)); so $\vec{x} = H(\vec{y})$ *is equivalent to* $g(\vec{x},\vec{y}) = \vec{0}$ for $(\vec{x},\vec{y}) \in P \times Q$.

Then, for all such $(\vec{x},\vec{y})$, with $\vec{x} = H(\vec{y})$, we surely have $g(\vec{x},\vec{y}) = \vec{0}$ and also

$$f(\vec{x},\vec{y}) = f(H(\vec{y}),\vec{y}) = f(\sigma(\vec{y})).$$

Set $h = f \circ \sigma$, $h\colon E^m \to E^1$. Then (10) reduces to

$$h(\vec{y}) - h(\vec{q}) \leq 0 \ (\text{or } \ \geq 0) \quad \text{for all } \vec{y} \in Q.$$

This means that h has an *unconditional* extremum at $\vec{q}$, an interior point of Q. Thus, by Theorem 1 in §9,

$$D_i h(\vec{q}) = 0, \quad i = 1, \ldots, m.$$

Hence, applying the chain rule (Theorem 2 of §4) to $h = f \circ \sigma$, we get, much as in the previous proof,

$$\begin{aligned} 0 &= \sum_{j=1}^{n+m} D_j f(\vec{p},\vec{q}) D_i \sigma_j(\vec{q}) \\ &= \sum_{j=1}^{n} D_j f(\vec{p},\vec{q}) D_i H_j(\vec{q}) + D_{n+i} f(\vec{p},\vec{q}), \quad i \leq m. \end{aligned} \tag{11}$$

(Verify!)

Next, as g by hypothesis satisfies Theorem 1, we get equations (3) or equivalently (6). Multiplying (6) by c_k, adding and combining with (11), we obtain

$$\sum_{j=1}^{n} \Big[D_j f(\vec{p},\vec{q}) + \sum_{k=1}^{n} c_k D_j g_k(\vec{p},\vec{q}) \Big] D_i H_j(\vec{q}) \\ + D_{n+i} f(\vec{p},\vec{q}) + \sum_{k=1}^{n} c_k D_{n+i} g_k(\vec{p},\vec{q}) = 0, \quad i \leq m.$$

(Verify!) But the square-bracketed expression is 0; for we chose the c_k so as to satisfy (9) for $j \le n$. Thus all simplifies to

$$\sum_{k=1}^{n} c_k D_{n+i} g_k(\vec{p}, \vec{q}) = -D_{n+i} f(\vec{p}, \vec{q}), \quad i = 1, 2, \ldots, m.$$

Hence (9) holds for $n < j \le n + m$, too, and all is proved. □

Remarks. Lagrange's method has the advantage that *all* variables (the x_k and y_i) are treated *equally*, without singling out the *dependent* ones. Thus in applications, one uses only F, i.e., f and g (*not* H).

One can also write $\vec{x} = (x_1, \ldots, x_{n+m})$ for $(\vec{x}, \vec{y}) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$ (the "double" notation was good for the *proof* only).

On the other hand, one still must *solve equations* (8).

Theorem 2 yields only a *necessary* condition (9) for extrema with constraints. There also are various *sufficient* conditions, but mostly one uses geometric and other considerations instead (as we did in §9). Therefore, we limit ourselves to *one* proposition (using "single" notation this time).

Theorem 3 (sufficient conditions). *Let*

$$F = f + \sum_{k=1}^{n} c_k g_k,$$

with $f\colon E^{n+m} \to E^1$, $g : E^{n+m} \to E^n$, *and* c_k *as in Theorem* 2.

Then f *has a maximum (minimum) at* $\vec{p} = (p_1, \ldots, p_{n+m})$ (*with constraints* $g = (g_1, \ldots, g_n) = \vec{0}$) *whenever* F *does. (A fortiori, this is the case if* F *has an* unconditional *extremum at* $\vec{p}$.)

Proof. Suppose F has a maximum at $\vec{p}$, with constraints $g = \vec{0}$. Then

$$0 \ge F(\vec{x}) - F(\vec{p}) = f(\vec{x}) - f(\vec{p}) + \sum_{k=1}^{n} c_k \left[g_k(\vec{x}) - g_k(\vec{p})\right]$$

for those $\vec{x}$ near $\vec{p}$ (including $\vec{x} = \vec{p}$) for which $g(\vec{x}) = \vec{0}$.

But for such $\vec{x}$, $g_k(\vec{x}) = g_k(\vec{p}) = 0$, $c_k \left[g_k(\vec{x}) - g_k(\vec{p})\right] = 0$, and so

$$0 \ge F(\vec{x}) - F(\vec{p}) = f(\vec{x}) - f(\vec{p}).$$

Hence f has a maximum at $\vec{p}$, with constraints as stated.

Similarly, $\Delta F = \Delta f$ in case F has a conditional *minimum* at $\vec{p}$. □

Example 1.

Find the local extrema of

$$f(x, y, z, t) = x + y + z + t$$

on the condition that

$$g(x, y, z, t) = xyzt - a^4 = 0,$$

with $a > 0$ and $x, y, z, t > 0$. (Note that *inequalities* do not count as "constraints" in the sense of Theorems 2 and 3.) Here one can simply eliminate $t = a^4/(xyz)$, but it is still easier to use Lagrange's method.

Set $F(x, y, z, t) = x + y + z + t + cxyzt$. (We drop a^4 since it will anyway disappear in differentiation.) Equations (8) then read

$$0 = 1 + cyzt = 1 + cxzt = 1 + cxyt = 1 + cxyz, \quad xyzt - a^4 = 0.$$

Solving for x, z, t and c, we get $c = -a^{-3}$, $x = y = z = t = a$.

Thus $F(x, y, z, t) = x + y + z + t - xyzt/a^3$, and the only critical point is $\vec{p} = (a, a, a, a)$. (Verify!)

By Theorem 3, one can now explore the sign of $F(\vec{x}) - F(\vec{p})$, where $\vec{x} = (x, y, z, t)$. For $\vec{x}$ near $\vec{p}$, it agrees with the sign of $d^2F(\vec{p}, \cdot\,)$. (See proof of Theorem 2 in §9.) We shall do it below, using yet another device, to be explained now.

Elimination of dependent differentials. If all partials of F vanish at $\vec{p}$ (e.g., if $\vec{p}$ satisfies (9)), then $d^1F(\vec{p}, \cdot\,) = 0$ on E^{n+m} (briefly $dF \equiv 0$).

Conversely, if $d^1f(\vec{p}, \cdot\,) = 0$ on a globe $G_{\vec{p}}$, for some function f on n *independent* variables, then

$$D_kf(\vec{p}) = 0, \quad k = 1, 2, \ldots, n,$$

since $d^1f(\vec{p}, \cdot\,)$ (a polynomial!) vanishes at infinitely many points if its coefficients $D_kf(\vec{p})$ vanish. (The latter fails, however, if the variables are *interdependent*.)

Thus, instead of working with the *partials*, one can equate to 0 the *differential* dF or df. Using the "variable" notation and the invariance of df (Note 4 in §4), one then writes $dx, dy, \ldots$ for the "differentials" of dependent and independent variables alike, and tries to eliminate the differentials of the *dependent* variables. We now redo Example 1 using this method.

Example 2.

With f and g as in Example 1, we treat t as the dependent variable, i.e., an implicit function of x, y, z,

$$t = a^4/(xyz) = H(x, y, z),$$

and differentiate the identity $xyzt - a^4 = 0$ to obtain

$$0 = yzt\,dx + xzt\,dy + xyt\,dz + xyz\,dt;$$

so

$$dt = -t\left(\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}\right). \tag{12}$$

Substituting this value of dt in $df = dx + dy + dz + dt = 0$ (the equation for critical points), we *eliminate* dt and find:

$$\left(1 - \frac{t}{x}\right) dx + \left(1 - \frac{t}{y}\right) dy + \left(1 - \frac{t}{z}\right) dz \equiv 0.$$

As x, y, z are *independent* variables, this identity implies that the coefficients of dx, dy, and dz must vanish, as pointed out above. Thus

$$1 - \frac{t}{x} = 1 - \frac{t}{y} = 1 - \frac{t}{z} = 0.$$

Hence $x = y = z = t = a$. (Why?) Thus again, the only critical point is $\vec{p} = (a, a, a, a)$.

Now, returning to Lagrange's method, we use formula (5) in §5 to compute

$$d^2F = -\frac{2}{a}(dx\,dy + dx\,dz + dz\,dt + dx\,dt + dy\,dz + dy\,dt). \tag{13}$$

(Verify!)

We shall show that this expression is *sign-constant* (if $xyzt = a^4$), near the critical point $\vec{p}$. Indeed, setting $x = y = z = t = a$ in (12), we get $dt = -(dx + dy + dz)$, and (13) turns into

$$-\frac{2}{a}\left[dx\,dy + dx\,dz + dy\,dz - (dx + dy + dz)^2\right]$$
$$= \frac{1}{a}\left[dx^2 + dy^2 + dz^2 + (dx + dy + dz)^2\right] = d^2F.$$

This expression is > 0 (for dx, dy, and dz are not *all* 0). Thus f has a *local* conditional minimum at $\vec{p} = (a, a, a, a)$.

Caution; here we cannot infer that $f(\vec{p})$ is the *least* value of f under the imposed conditions: $x, y, z > 0$ and $xyzt = a^4$.

The simplification due to the Cauchy invariant rule (Note 4 in §4) makes the use of the "variable" notation attractive, though caution is mandatory.

Note 2. When using Theorem 2, it suffices to ascertain that *some* n equations from (9) admit a solution for the c_k; for then, renumbering the equations, one can achieve that these become the *first* n equations, as was assumed. This means that the $n \times (n + m)$ matrix $(D_j g_k(\vec{p}, \vec{q}))$ must be *of rank* n, i.e., *contains an* $n \times n$*-submatrix* (*obtained by deleting some columns*), *with a nonzero determinant*.

In the Problems we often use $r, s, t, \ldots$ for Lagrange multipliers.

Further Problems on Maxima and Minima

1. Fill in all details in Examples 1 and 2 and the proofs of all theorems in this section.

2. Redo Example (B) in §9 by Lagrange's method.
[Hint: Set $F(x, y, z) = f(x, y, z) - r(x^2 + y^2 + z^2)$, $g(x, y, z) = x^2 + y^2 + z^2 - 1$. Compare the values of f at all critical points.[2]]

3. An ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is cut by a plane $ux + vy + wz = 0$. Find the semiaxes of the section-ellipse, i.e., the extrema of

$$\rho^2 = [f(x, y, z)]^2 = x^2 + y^2 + z^2$$

under the constraints $g = (g_1, g_2) = \vec{0}$, where

$$g_1(x, y, z) = ux + vy + wz \text{ and } g_2(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

Assume that $a > b > c > 0$ and that not all $u, v, w = 0$.
[Outline: By Note 2, explore the rank of the matrix

$$\begin{pmatrix} x/a^2 & y/b^2 & z/c^2 \\ u & v & z \end{pmatrix}. \tag{14}$$

(Why *this* particular matrix?)

Seeking a contradiction, suppose all its 2×2 determinants vanish at all points of the section-ellipse. Then the upper and lower entries in (14) are *proportional* (why?); so $x^2/a^2 + y^2/b^2 + z^2/c^2 = 0$ (a contradiction!).

Next, set

$$F(x, y, z) = x^2 + y^2 + z^2 + r\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + 2s(ux + vy + wz).$$

Equate dF to 0:

$$x + \frac{rx}{a^2} + su = 0, \quad y + \frac{ry}{b^2} + sv = 0, \quad z + \frac{rz}{c^2} + sw = 0. \tag{15}$$

Multiplying by x, y, z, respectively, adding, and combining with $g = \vec{0}$, obtain $r = -\rho^2$; so, by (15), for $a, b, c \neq \rho$,

$$x = \frac{-sua^2}{a^2 - \rho^2}, \quad y = \frac{-svb^2}{b^2 - \rho^2}, \quad z = \frac{-swc^2}{c^2 - \rho^2}.$$

Find s, x, y, z, then compare the ρ-values at critical points.]

---

[2] This suffices here, since the equation $g = \vec{0}$ defines a *compact* set S; see §9.

4. Find the least and the largest values of the quadratic form

$$f(\vec{x}) = \sum_{i,k=1}^{n} a_{ik} x_i x_k \quad (a_{ik} = a_{ki})$$

on the condition that $g(\vec{x}) = |\vec{x}|^2 - 1 = 0$ ($f, g \colon E^n \to E^1$).
[Outline: Let $F(\vec{x}) = f(\vec{x}) - t(x_1^2 + x_2^2 + \ldots + x_n^2)$. Equating dF to 0, obtain

$$\begin{aligned} (a_{11} - t)\, x_1 + a_{12}\, x_2 + \ldots + a_{1n}\, x_n &= 0, \\ a_{21}\, x_1 + (a_{22} - t)\, x_2 + \ldots + a_{2n}\, x_n &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}\, x_1 + a_{n2}\, x_2 + \ldots + (a_{nn} - t)\, x_n &= 0. \end{aligned} \tag{16}$$

Using Theorem 1(iv) in §6, derive the so-called *characteristic equation* of f,

$$\begin{vmatrix} a_{11} - t & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - t & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{2n} & \ldots & a_{nn} - t \end{vmatrix} = 0, \tag{17}$$

of degree n in t. If t is one of its n roots (*known* to be real[3]), then equations (16) admit a nonzero solution for $\vec{x} = (x_1, \ldots, x_n)$; by replacing $\vec{x}$ by $\vec{x}/|\vec{x}|$ if necessary, $\vec{x}$ satisfies also the constraint equation $g(\vec{x}) = |\vec{x}|^2 - 1 = 0$. (Explain!) Thus each root t of (17) yields a critical point $\vec{x}_t = (x_1, \ldots, x_n)$.

Now, to find $f(\vec{x}_t)$, multiply the kth equation in (16) by x_k, $k = 1, \ldots, n$, and add to get

$$0 = \sum_{i,k=1}^{n} a_{ik} x_i x_k - t \sum_{k=1}^{n} x_k^2 = f(\vec{x}_t) - t.$$

Hence $f(\vec{x}_t) = t$.

Thus the values of f at the critical points $\vec{x}_t$ are simply the roots of (17). The largest (smallest) root is also the largest (least) value of f on $S = \{\vec{x} \in E^n \mid |\vec{x}| = 1\}$. (Explain!)]

5. Use the method of Problem 4 to find the semiaxes of

(i) the quadric curve in E^2, centered at $\vec{0}$, given by $\sum_{i,k=1}^{2} a_{ik} x_i x_k = 1$; and

(ii) the quadric surface $\sum_{i,k=1}^{3} a_{ik} x_i x_k = 1$ in E^3, centered at $\vec{0}$.

Assume $a_{ik} = a_{ki}$.
[Hint: Explore the extrema of $f(\vec{x}) = |\vec{x}|^2$ on the condition that

$$g(\vec{x}) = \sum_{i,k} a_{ik} x_i x_k - 1 = 0.]$$

6. Using Lagrange's method, redo Problems 4, 5, 6, 7, 11, 12, and 13 of §9.

7. In E^2, find the shortest distance from $\vec{0}$ to the parabola $y^2 = 2(x + a)$.

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[3] See S. Perlis, *Theory of Matrices*, Reading, Mass., 1952, Theorem 9-25.

8. In E^3, find the shortest distance from $\vec{0}$ to the intersection line of two planes given by the formulas $\vec{u} \cdot \vec{x} = a$ and $\vec{v} \cdot \vec{x} = b$ with $\vec{u}$ and $\vec{v}$ different from $\vec{0}$. (Rewrite all in coordinate form!)

9. In E^n, find the largest value of $|\vec{a} \cdot \vec{x}|$ if $|\vec{x}| = 1$. Use Lagrange's method.

***10.** (Hadamard's theorem.) If $A = \det(x_{ik})$ $(i, k \leq n)$, then

$$|A| \leq \prod_{i=1}^{n} |\vec{x}_i|,$$

where $\vec{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})$.

[Hints: Set $a_i = |\vec{x}_i|$. Treat A as a function of n^2 variables. Using Lagrange's method, prove that, under the n constraints $|\vec{x}_i|^2 - a_i^2 = 0$, A cannot have an extremum unless $A^2 = \det(y_{ik})$, with $y_{ik} = 0$ (if $i \neq k$) and $y_{ii} = a_i^2$.]

Chapter 7

Volume and Measure

Our intuitive idea of “volume” is rather vague. We just tend to assume that “bodies” in space (i.e., in E^3) somehow have numerically expressed “volumes,” but it remains unclear which sets in E^3 are “bodies” and how volume is *defined*.

We also intuitively assume that volumes behave “additively.” That is, if a body is split into disjoint parts, then the volume of the whole equals the sum of the volumes of the parts. Similarly for “areas” in E^2. In elementary calculus, that is often just taken for granted.

The famous mathematician Henri Lebesgue (1875–1941) extended the idea of “volume” to a large, strictly defined family of sets in E^n, called *Lebesgue-measurable* sets, thus giving rise to what is called *measure theory*. Its basic idea remains that of additivity, precisely formulated and *proved*. Modern theory has still more generalized these ideas. In this text, we have so far defined “volumes” for intervals in E^n only. Thus it is natural to take intervals as our starting point. This will also lead to the important idea of a *semiring of sets* and its extension: *a ring of sets*.

§1. More on Intervals in E^n. Semirings of Sets

I. As a prologue, we turn to intervals in E^n (Chapter 3, §7).

Theorem 1. *If A and B are intervals in E^n, then*

(i) *$A \cap B$ is an interval* (*$\emptyset$ counts as an interval*);

(ii) *$A - B$ is the union of finitely many disjoint intervals* (*but need not be an interval itself*).

Proof. The easy proof *for* E^1 is left to the reader.

An interval *in* E^2 is the cross-product of two line intervals.

Let

$$A = X \times Y \text{ and } B = X' \times Y',$$

where X, Y, X', and Y' are intervals in E^1. Then (see Figure 29)

$$A \cap B = (X \times Y) \cap (X' \times Y') = (X \cap X') \times (Y \cap Y')$$

and

$$A - B = [(X - X') \times Y] \cup [(X \cap X') \times (Y - Y')];$$

see Problem 8 in Chapter 1, §§1–3.

As the theorem holds in E^1,

$$X \cap X' \text{ and } Y \cap Y'$$

are intervals in E^1, while

$$X - Y' \text{ and } Y - Y'$$

are finite unions of disjoint line intervals. (In Figure 29 they are just intervals, but in general they are not.)

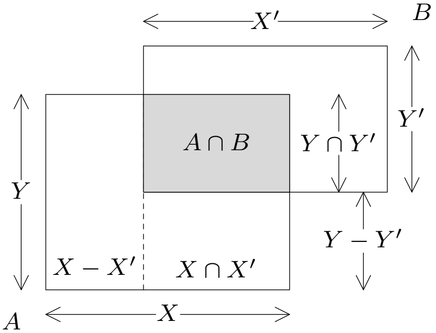

FIGURE 29

It easily follows that $A \cap B$ is an interval in E^2, while $A - B$ splits into finitely many such intervals. (Verify!) Thus the theorem holds in E^2.

Finally, for E^n, use induction. An interval in E^n is the cross-product of an interval in E^{n-1} by a line interval. Thus if the theorem holds in E^{n-1}, *the same* argument shows that it holds in E^n, too. (Verify!)

This completes the inductive proof. □

Actually, Theorem 1 applies to many other families of sets (not necessarily intervals or sets in E^n). We now give such families a name.

Definition 1.

A family $\mathcal{C}$ of arbitrary sets is called a *semiring* iff

(i) $\emptyset \in \mathcal{C}$ ($\emptyset$ is a member), and

(ii) for any sets A and B from $\mathcal{C}$, we have $A \cap B \in \mathcal{C}$, while $A - B$ is the union of finitely many disjoint sets from $\mathcal{C}$.

Briefly: *$\mathcal{C}$ is a semiring iff it satisfies Theorem* 1.

Note that here $\mathcal{C}$ is not just a set, but a whole *family of sets*. Recall (Chapter 1, §§1–3) that a *set family* (family of sets) is a set $\mathcal{M}$ whose members are other sets. If A is a member of $\mathcal{M}$, we call A an $\mathcal{M}$-set and write $A \in \mathcal{M}$ (not $A \subseteq \mathcal{M}$).

Sometimes we use *index notation*:

$$\mathcal{M} = \{X_i \mid i \in I\},$$

briefly

$$\mathcal{M} = \{X_i\},$$

where the X_i are $\mathcal{M}$-sets distinguished from each other by the subscripts i varying over some index set I.

A set family $\mathcal{M} = \{X_i\}$ and its union

$$\bigcup_i X_i$$

are said to be *disjoint* iff

$$X_i \cap X_j = \emptyset \text{ whenever } i \neq j.$$

Notation:

$$\bigcup X_i \ (disjoint).$$

In our case, $A \in \mathcal{C}$ means that A is a $\mathcal{C}$*-set* (a member of the semiring $\mathcal{C}$). The formula

$$(\forall A, B \in \mathcal{C}) \quad A \cap B \in \mathcal{C}$$

means that the intersection of two $\mathcal{C}$-sets in a $\mathcal{C}$-set itself.

Henceforth, we will often speak of semirings $\mathcal{C}$ *in general*. In particular, this will apply to the case $\mathcal{C} = \{\text{intervals}\}$. Always keep this case in mind!

Note 1. By Theorem 1, *the intervals in E^n form a semiring*. So also do the *half-open* and the *half-closed* intervals separately (same proof!), but not the open (or closed) ones. (Why?)

Caution. The union and difference of two $\mathcal{C}$-sets *need not* be a $\mathcal{C}$-set. To remedy this, we now *enlarge* $\mathcal{C}$.

Definition 2.

We say that a set A (from $\mathcal{C}$ or not) *is* $\mathcal{C}$*-simple* and write

$$A \in \mathcal{C}_s'$$

iff A is a finite union of disjoint $\mathcal{C}$-sets (such as $A - B$ in Theorem 1).

Thus $\mathcal{C}_s'$ *is the family of all* $\mathcal{C}$*-simple sets*.

Every $\mathcal{C}$-set is also a $\mathcal{C}_s'$-set, i.e., a $\mathcal{C}$-simple one. (Why?) Briefly:

$$\mathcal{C} \subseteq \mathcal{C}_s'.$$

If $\mathcal{C}$ is the set of all intervals, a $\mathcal{C}$-simple set may look as in Figure 30.

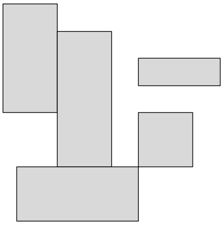

FIGURE 30

Theorem 2. *If $\mathcal{C}$ is a semiring, and if A and B are $\mathcal{C}$-simple, so also are*

$$A \cap B,\ A - B,\ and\ A \cup B.$$

In symbols,

$$(\forall A, B \in \mathcal{C}_s') \quad A \cap B \in \mathcal{C}_s',\ A - B \in \mathcal{C}_s',\ and\ A \cup B \in \mathcal{C}_s'.$$

We give a *proof outline* and suggest the proof as an exercise. Before attempting it, the reader should thoroughly review the laws and problems of Chapter 1, §§1-3.

(1) To prove $A \cap B \in \mathcal{C}_s'$, let

$$A = \bigcup_{i=1}^{m} A_i \ (\textit{disjoint}) \text{ and } B = \bigcup_{k=1}^{n} B_k \ (\textit{disjoint}),$$

with $A_i, B_k \in \mathcal{C}$. Verify that

$$A \cap B = \bigcup_{k=1}^{n} \bigcup_{i=1}^{m} (A_i \cap B_k) \ (\textit{disjoint}),$$

and so $A \cap B \in \mathcal{C}_s'$.

(2) Next prove that $A - B \in \mathcal{C}_s'$ if $A \in \mathcal{C}_s'$ and $B \in \mathcal{C}$.

Indeed, if

$$A = \bigcup_{i=1}^{m} A_i \ (\textit{disjoint}),$$

then

$$A - B = \bigcup_{i=1}^{m} A_i - B = \bigcup_{i=1}^{m} (A_i - B) \ (\textit{disjoint}).$$

Verify and use Definition 2.

(3) Prove that

$$(\forall\, A, B \in \mathcal{C}_s') \quad A - B \in \mathcal{C}_s';$$

we suggest the following argument.

Let

$$B = \bigcup_{k=1}^{n} B_k, \quad B_k \in \mathcal{C}.$$

Then

$$A - B = A - \bigcup_{k=1}^{n} B_k = \bigcap_{k=1}^{n} (A - B_k)$$

by duality laws. But $A - B_k$ is $\mathcal{C}$-simple by step (2). Hence so is

$$A - B = \bigcap_{k=1}^{m} (A - B_k)$$

by step (1) plus induction.

(4) To prove $A \cup B \in \mathcal{C}_s'$, verify that

$$A \cup B = A \cup (B - A),$$

where $B - A \in \mathcal{C}_s'$, by (3).

Note 2. By induction, Theorem 2 extends to any finite number of $\mathcal{C}_s'$-sets. It is a kind of "closure law."

We thus briefly say that $\mathcal{C}_s'$ *is closed under finite unions, intersections, and set differences.* Any (nonempty) set family with these properties is called a *set ring* (see also §3).

Thus Theorem 2 states that *if* $\mathcal{C}$ *is a semiring, then* $\mathcal{C}_s'$ *is a ring.*

Caution. An *infinite* union of $\mathcal{C}$-simple sets need not be $\mathcal{C}$-simple. Yet we may consider such unions, as we do next.

In Corollary 1 below, $\mathcal{C}_s'$ may be replaced by *any* set ring $\mathcal{M}$.

Corollary 1. *If* $\{A_n\}$ *is a finite or infinite sequence of sets from a semiring* $\mathcal{C}$ (*or from a ring* $\mathcal{M}$ *such as* $\mathcal{C}_s'$), *then there is a* disjoint *sequence of* $\mathcal{C}$*-simple sets* (*or* $\mathcal{M}$*-sets*) $B_n \subseteq A_n$ *such that*

$$\bigcup_n A_n = \bigcup_n B_n.$$

Proof. Let $B_1 = A_1$ and for $n = 1, 2, \ldots,$

$$B_{n+1} = A_{n+1} - \bigcup_{k=1}^{n} A_k, \quad A_k \in \mathcal{C}.$$

By Theorem 2, the B_n are $\mathcal{C}$-simple (as are A_{n+1} and $\bigcup_{k=1}^{n} A_k$). Show that they are disjoint (assume the opposite and find a contradiction) and verify that $\bigcup A_n = \bigcup B_n$: If $x \in \bigcup A_n$, take the *least* n for which $x \in A_n$. Then $n > 1$ and

$$x \in A_n - \bigcup_{k=1}^{n-1} A_k = B_n,$$

or $n = 1$ and $x \in A_1 = B_1$. □

Note 3. In Corollary 1, $B_n \in \mathcal{C}_s'$, i.e., $B_n = \bigcup_{i=1}^{m_n} C_{ni}$ for some *disjoint* sets $C_{ni} \in \mathcal{C}$. Thus

$$\bigcup_n A_n = \bigcup_n \bigcup_{i=1}^{m_n} C_{ni}$$

is also a countable disjoint union of $\mathcal{C}$-sets.

II. Recall that the volume of intervals is *additive* (Problem 9 in Chapter 3, §7). That is, if $A \in \mathcal{C}$ is split into finitely many disjoint subintervals, then vA (the volume of A) equals the sum of the volumes of the parts.

We shall need the following lemma.

Lemma 1. *Let* $X_1, X_2, \ldots, X_m \in \mathcal{C}$ *(intervals in* E^n*). If the* X_i *are mutually disjoint, then*

(i) $\bigcup_{i=1}^{m} X_i \subseteq Y \in \mathcal{C}$ *implies* $\sum_{i=1}^{m} vX_i \le vY$*; and*

(ii) $\bigcup_{i=1}^{m} X_i \subseteq \bigcup_{k=1}^{p} Y_k$ *(with* $Y_k \in \mathcal{C}$*) implies* $\sum_{i=1}^{m} vX_i \le \sum_{k=1}^{p} vY_k$.

Proof. (i) By Theorem 2, the set

$$Y - \bigcup_{i=1}^{m} X_i$$

is $\mathcal{C}$-simple; so

$$Y - \bigcup_{i=1}^{m} X_i = \bigcup_{j=1}^{q} C_j$$

for some disjoint intervals C_j. Hence

$$Y = \bigcup X_i \cup \bigcup C_j \ (\textit{all} \text{ disjoint}).$$

Thus by additivity,

$$vY = \sum_{i=1}^{m} vX_i + \sum_{j=1}^{q} vC_j \ge \sum_{i=1}^{m} vX_i,$$

as claimed.

(ii) By set theory (Problem 9 in Chapter 1, §§1–3),

$$X_i \subseteq \bigcup_{k=1}^{p} Y_k$$

implies

$$X_i = X_i \cap \bigcup_{k=1}^{p} Y_k = \bigcup_{k=1}^{p} (X_i \cap Y_k).$$

If it happens that the Y_k are mutually disjoint also, so certainly are the *smaller* intervals $X_i \cap Y_k$; so by additivity,

$$vX_i = \sum_{k=1}^{p} v(X_i \cap Y_k).$$

Hence

$$\sum_{i=1}^{m} vX_i = \sum_{i=1}^{m}\sum_{k=1}^{p} v(X_i \cap Y_k) = \sum_{k=1}^{p}\left[\sum_{i=1}^{m} v(X_i \cap Y_k)\right].$$

But by (i),

$$\sum_{i=1}^{m} v(X_i \cap Y_k) \le vY_k \text{ (why?);}$$

so

$$\sum_{i=1}^{m} vX_i \le \sum_{k=1}^{p} vY_k,$$

as required.

If, however, the Y_k are not disjoint, Corollary 1 yields

$$\bigcup Y_k = \bigcup B_k \ (\textit{disjoint}),$$

with

$$Y_k \supseteq B_k = \bigcup_{j=1}^{m_k} C_{kj} \ (\textit{disjoint}), \quad C_{kj} \in \mathcal{C}.$$

By (i),

$$\sum_{j=1}^{m_k} vC_{kj} \le vY_k.$$

As

$$\bigcup_{i=1}^{m} X_i \subseteq \bigcup_{k=1}^{p} Y_k = \bigcup_{k=1}^{p} B_k = \bigcup_{k=1}^{p}\bigcup_{j=1}^{m_k} C_{kj} \ (\textit{disjoint}),$$

all reduces to the previous disjoint case. □

Corollary 2. *Let* $A \in \mathcal{C}_s'$ ($\mathcal{C}$ = *intervals in* E^n). *If*

$$A = \bigcup_{i=1}^{m} X_i \ (\textit{disjoint}) = \bigcup_{k=1}^{p} Y_k \ (\textit{disjoint})$$

with $X_i, Y_k \in \mathcal{C}$, *then*

$$\sum_{i=1}^{m} vX_i = \sum_{k=1}^{p} vY_k.$$

(Use part (ii) of the lemma twice.)

Thus we can (and do) *unambiguously* define vA to be either of these sums.

Problems on Intervals and Semirings

1. Complete the proof of Theorem 1 and Note 1.

1′. Prove Theorem 2 in detail.

2. Fill in the details in the proof of Corollary 1.

2′. Prove Corollary 2.

3. Show that, in the definition of a semiring, the condition $\emptyset \in \mathcal{C}$ is equivalent to $\mathcal{C} \neq \emptyset$.
[Hint: Consider $\emptyset = A - A = \bigcup_{i=1}^{m} A_i$ $(A, A_i \in \mathcal{C})$ to get $\emptyset = A_i \in \mathcal{C}$.]

4. Given a set S, show that the following are semirings or rings.
  (a) $\mathcal{C} = \{\text{all subsets of } S\}$;
  (b) $\mathcal{C} = \{\text{all } \textit{finite} \text{ subsets of } S\}$;
  (c) $\mathcal{C} = \{\emptyset\}$;
  (d) $\mathcal{C} = \{\emptyset \text{ and all singletons in } S\}$.

Disprove it for $\mathcal{C} = \{\emptyset \text{ and all } \textit{two-point} \text{ sets in } S\}$, $S = \{1, 2, 3, \ldots\}$. In (a)–(c), show that $\mathcal{C}_s' = \mathcal{C}$. Disprove it for (d).

5. Show that the *cubes* in E^n $(n > 1)$ do not form a semiring.

6. Using Corollary 2 and the definition thereafter, show that volume is additive for $\mathcal{C}$-*simple sets*. That is,

$$\text{if } A = \bigcup_{i=1}^{m} A_i \ (\textit{disjoint}) \text{ then } vA = \sum_{i=1}^{m} vA_i \quad (A, A_i \in \mathcal{C}_s').$$

7. Prove the lemma for $\mathcal{C}$-*simple* sets.
[Hint: Use Problem 6 and argue as before.]

8. Prove that if $\mathcal{C}$ is a semiring, then $\mathcal{C}_s'$ ($\mathcal{C}$-simple sets) $= \mathcal{C}_s$, the family of *all* finite unions of $\mathcal{C}$-sets (disjoint or not).
[Hint: Use Theorem 2.]

§2. $\mathcal{C}_\sigma$-Sets. Countable Additivity. Permutable Series

We now want to further extend the definition of volume by considering *countable* unions of intervals, called $\mathcal{C}_\sigma$-sets ($\mathcal{C}$ being the semiring of all intervals in E^n).

We also ask, if A is split into *countably many* such sets, does additivity still hold? This is called *countable additivity* or σ-*additivity* (the σ is used whenever *countable* unions are involved).

We need two lemmas in addition to that of §1.

Lemma 1. *If B is a nonempty interval in E^n, then given $\varepsilon > 0$, there is an open interval C and a closed one A such that*

$$A \subseteq B \subseteq C$$

and

$$vC - \varepsilon < vB < vA + \varepsilon.$$

Proof. Let the endpoints of B be

$$\bar{a} = (a_1, \ldots, a_n) \text{ and } \bar{b} = (b_1, \ldots, b_n).$$

For each natural number i, consider the open interval C_i, with endpoints

$$\left(a_1 - \frac{1}{i}, a_2 - \frac{1}{i}, \ldots, a_n - \frac{1}{i}\right) \text{ and } \left(b_1 + \frac{1}{i}, b_2 + \frac{1}{i}, \ldots, b_n + \frac{1}{i}\right).$$

Then $B \subseteq C_i$ and

$$vC_i = \prod_{k=1}^{n} \left[b_k + \frac{1}{i} - \left(a_k - \frac{1}{i}\right)\right] = \prod_{k=1}^{n} \left(b_k - a_k + \frac{2}{i}\right).$$

Making $i \to \infty$, we get

$$\lim_{i\to\infty} vC_i = \prod_{k=1}^{n} (b_k - a_k) = vB.$$

(Why?) Hence by the sequential limit definition, given $\varepsilon > 0$, there is a natural i such that

$$vC_i - vB < \varepsilon,$$

or

$$vC_i - \varepsilon < vB.$$

As C_i is open and $\supseteq B$, it is the desired interval C.

Similarly, one finds the closed interval $A \subseteq B$. (Verify!) □

Lemma 2. *Any open set $G \subseteq E^n$ is a countable union of open cubes A_k and also a* disjoint *countable union of half-open intervals.*

(See also Problem 2 below.)

Proof. If $G = \emptyset$, take all $A_k = \emptyset$.

If $G \neq \emptyset$, every point $p \in G$ has a cubic neighborhood

$$C_p \subseteq G,$$

centered at p (Problem 3 in Chapter 3, §12). By slightly shrinking this C_p, one can make its endpoints *rational*, with p still in it (but not necessarily its center), and make C_p open, half-open, or closed, as desired. (Explain!)

Choose such a cube C_p for *every* $p \in G$; so

$$G \subseteq \bigcup_{p \in G} C_p.$$

But by construction, G contains *all* C_p, so that

$$G = \bigcup_{p \in G} C_p.$$

Moreover, because the coordinates of the endpoints of all C_p are *rational*, the set of ordered pairs of endpoints of the C_p is *countable*, and thus, while the set of all $p \in G$ is *uncountable*, the set of *distinct* C_p is countable. Thus one can put the family of all C_p in a sequence and rename it $\{A_k\}$:

$$G = \bigcup_{k=1}^{\infty} A_k.$$

If, further, the A_k are *half-open*, we can use Corollary 1 and Note 3, both from §1, to make the union *disjoint* (half-open intervals form a semiring!). □

Now let $\mathcal{C}_\sigma$ be the family of *all possible* countable unions of intervals in E^n, such as G in Lemma 2 (we use $\mathcal{C}_s$ for all *finite* unions). Thus $A \in \mathcal{C}_\sigma$ means that A is a $\mathcal{C}_\sigma$-set, i.e.,

$$A = \bigcup_{i=1}^{\infty} A_i$$

for some sequence of intervals $\{A_i\}$. Such are all open sets in E^n, but there also are many other $\mathcal{C}_\sigma$-sets.

We can always make the sequence $\{A_i\}$ *infinite* (add null sets or repeat a term!).

By Corollary 1 and Note 3 of §1, we can decompose any $\mathcal{C}_\sigma$-set A into countably many *disjoint* intervals. This can be done in many ways. However, we have the following result.

Theorem 1. *If*

$$A = \bigcup_{i=1}^{\infty} A_i \ (\textit{disjoint}) = \bigcup_{k=1}^{\infty} B_k \ (\textit{disjoint})$$

for some intervals A_i, B_k *in* E^n, *then*

$$\sum_{i=1}^{\infty} vA_i = \sum_{k=1}^{\infty} vB_k.^{1}$$

---

[1] Recall that a positive series *always* has a (possibly infinite) sum.

Thus we can (and do) *unambiguously* define either of these sums to be the *volume* vA of the $\mathcal{C}_\sigma$-set A.

Proof. We shall use the Heine–Borel theorem (Problem 10 in Chapter 4, §6; review it!).

Seeking a contradiction, let (say)

$$\sum_{i=1}^{\infty} vA_i > \sum_{k=1}^{\infty} vB_k,$$

so, in particular,

$$\sum_{k=1}^{\infty} vB_k < +\infty.$$

As

$$\sum_{i=1}^{\infty} vA_i = \lim_{m\to\infty} \sum_{i=1}^{m} vA_i,$$

there is an integer m for which

$$\sum_{i=1}^{m} vA_i > \sum_{k=1}^{\infty} vB_k.$$

We fix that m and set

$$2\varepsilon = \sum_{i=1}^{m} vA_i - \sum_{k=1}^{\infty} vB_k > 0.$$

Dropping "empties" (if any), we assume $A_i \neq \emptyset$ and $B_k \neq \emptyset$.

Then Lemma 1 yields open intervals $Y_k \supseteq B_k$, with

$$vB_k > vY_k - \frac{\varepsilon}{2^k}, \quad k = 1, 2, \dots\,,$$

and closed ones $X_i \subseteq A_i$, with

$$vX_i + \frac{\varepsilon}{m} > vA_i;$$

so

$$\begin{aligned}2\varepsilon = \sum_{i=1}^{m} vA_i - \sum_{k=1}^{\infty} vB_k &< \sum_{i=1}^{m}\Big(vX_i + \frac{\varepsilon}{m}\Big) - \sum_{k=1}^{\infty}\Big(vY_k - \frac{\varepsilon}{2^k}\Big)\\ &= \sum_{i=1}^{m} vX_i - \sum_{k=1}^{\infty} vY_k + 2\varepsilon.\end{aligned}$$

Thus

$$\sum_{i=1}^{m} vX_i > \sum_{k=1}^{\infty} vY_k. \tag{1}$$

(Explain in detail!)

Now, as

$$X_i \subseteq A_i \subseteq A = \bigcup_{k=1}^{\infty} B_k \subseteq \bigcup_{k=1}^{\infty} Y_k,$$

each of the *closed* intervals X_i is covered by the *open* sets Y_k.

By the Heine–Borel theorem, $\bigcup_{i=1}^{m} X_i$ is already covered by a *finite* number of the Y_k, say,

$$\bigcup_{i=1}^{m} X_i \subseteq \bigcup_{k=1}^{p} Y_k.$$

The X_i are *disjoint*, for even the larger sets A_i are. Thus by Lemma 1(ii) in §1,

$$\sum_{i=1}^{m} vX_i \le \sum_{k=1}^{p} vY_k \le \sum_{k=1}^{\infty} vY_k,$$

contrary to (1). This contradiction completes the proof. □

Corollary 1. *If*

$$A = \bigcup_{k=1}^{\infty} B_k \ (\textit{disjoint})$$

for some intervals B_k, *then*

$$vA = \sum_{k=1}^{\infty} vB_k.$$

Indeed, this is simply the definition of vA contained in Theorem 1.

Note 1. In particular, Corollary 1 holds if A is an interval itself. We express this by saying that the volume of intervals is σ*-additive* or *countably additive*. This also shows that our previous definition of volume (for intervals) agrees with the definition contained in Theorem 1 (for $\mathcal{C}_\sigma$-sets).

Note 2. As all open sets are $\mathcal{C}_\sigma$-sets (Lemma 2), volume is now defined for any open set $A \subseteq E^n$ (in particular, for $A = E^n$).

Corollary 2. *If* A_i, B_k *are intervals in* E^n, *with*

$$\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{k=1}^{\infty} B_k,$$

then provided the A_i *are mutually disjoint*,

$$\sum_{i=1}^{\infty} vA_i \le \sum_{k=1}^{\infty} vB_k. \tag{2}$$

The proof is as in Theorem 1 (but the B_k need not be disjoint here).

Corollary 3 ("σ-subadditivity"[2] of the volume)**.** *If*

$$A \subseteq \bigcup_{k=1}^{\infty} B_k,$$

where $A \in \mathcal{C}_\sigma$ *and the* B_k *are intervals in* E^n, *then*

$$vA \leq \sum_{k=1}^{\infty} vB_k.$$

Proof. Set

$$A = \bigcup_{i=1}^{\infty} A_i \ (\textit{disjoint}),\ A_i \in \mathcal{C},$$

and use Corollary 2. □

Corollary 4 ("monotonicity"[2])**.** *If* $A, B \in \mathcal{C}_\sigma$, *with*

$$A \subseteq B,$$

then

$$vA \leq vB.$$

("*Larger sets have larger volumes.*")

This is simply Corollary 3, with $\bigcup_k B_k = B$.

Corollary 5. *The volume of all of* E^n *is* ∞ (*we write* ∞ *for* $+\infty$).

Proof. We have $A \subseteq E^n$ for any interval A.

Thus, by Corollary 4, $vA \leq vE^n$.

As vA can be chosen arbitrarily large, vE^n must be infinite. □

Corollary 6. *For any* countable *set* $A \subset E^n$, $vA = 0$. *In particular*, $v\emptyset = 0$.

Proof. First let $A = \{\bar{a}\}$ be a *singleton*. Then we may treat A as a degenerate interval $[\bar{a}, \bar{a}]$. As all its edge lengths are 0, we have $vA = 0$.

Next, if $A = \{\bar{a}_1, \bar{a}_2, \dots\}$ is a countable set, then

$$A = \bigcup_k \{\bar{a}_k\};$$

so

$$vA = \sum_k v\{\bar{a}_k\} = 0$$

by Corollary 1.

---

[2] This notion is treated in more detail in §5.

Finally, $\emptyset$ is the degenerate *open* interval $(\bar{a}, \bar{a})$; so $v\emptyset = 0$. $\square$

Note 3. Actually, all these propositions hold also if *all* sets involved are $\mathcal{C}_\sigma$-sets, not just intervals (split each $\mathcal{C}_\sigma$-set into disjoint intervals!).

Permutable Series. Since σ-additivity involves *countable sums*, it appears useful to generalize the notion of a series.

We say that a series of constants,

$$\sum a_n,$$

is *permutable* iff it has a definite (possibly infinite) sum obeying the *general commutative law*:

Given any one–one map

$$u\colon N \overset{\text{onto}}{\longleftrightarrow} N$$

($N =$ *the naturals*), *we have*

$$\sum_n a_n = \sum_n a_{u_n},$$

where $u_n = u(n)$.

(Such are all positive and all absolutely convergent series in a *complete* space E; see Chapter 4, §13.) If the series is permutable, the sum does not depend on the choice of the map u.

Thus, given any $u\colon N \overset{\text{onto}}{\longleftrightarrow} J$ (where J is a countable index set) and a set

$$\{a_i \mid i \in J\} \subseteq E$$

(where E is E^* or a normed space), we can define

$$\sum_{i \in J} a_i = \sum_{n=1}^{\infty} a_{u_n}$$

if $\sum_n a_{u_n}$ *is permutable.*

In particular, if

$$J = N \times N$$

(a *countable* set, by Theorem 1 in Chapter 1, §9), we call

$$\sum_{i \in J} a_i$$

a *double series*, denoted by symbols like

$$\sum_{n,k} a_{kn} \quad (k, n \in N).$$

Note that

$$\sum_{i\in J}|a_i|$$

is *always* defined (being a *positive* series).

If

$$\sum_{i\in J}|a_i|<\infty,$$

we say that $\sum_{i\in J}a_i$ *converges absolutely.*

For a positive series, we obtain the following result.

Theorem 2.

(i) *All* positive *series in* E^* *are permutable.*

(ii) *For positive double series in* E^*, *we have*

$$\sum_{n,k=1}^{\infty}a_{nk}=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}a_{nk}\right)=\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}a_{nk}\right). \tag{3}$$

Proof. (i) Let

$$s=\sum_{n=1}^{\infty}a_n \text{ and } s_m=\sum_{n=1}^{m}a_n \quad (a_n\geq 0).$$

Then clearly

$$s_{m+1}=s_m+a_{m+1}\geq s_m;$$

i.e., $\{s_m\}\uparrow$, and so

$$s=\lim_{m\to\infty}s_m=\sup_m s_m$$

by Theorem 3 in Chapter 3, §15.

Hence s certainly does not exceed the lub of *all possible* sums of the form

$$\sum_{i\in I}a_i,$$

where I is a *finite subset* of N (the partial sums s_m are *among* them). Thus

$$s\leq\sup\sum_{i\in I}a_i, \tag{4}$$

over all finite sets $I\subset N$.

On the other hand, every such $\sum_{i\in I}a_i$ is exceeded by, or equals, some s_m. Hence in (4), the *reverse* inequality holds, too, and so

$$s=\sup\sum_{i\in I}a_i.$$

But $\sup \sum_{i\in I} a_i$ clearly *does not depend on any arrangement of the* a_i. Therefore, the series $\sum a_n$ is permutable, and assertion (i) is proved.

Assertion (ii) follows similarly by considering sums of the form $\sum_{i\in I} a_i$, where I is a finite subset *of* $N \times N$, and showing that the lub of such sums equals each of the three expressions in (3). We leave it to the reader. □

A similar formula holds for *absolutely convergent* series (see Problems).

Problems on $\mathcal{C}_\sigma$-Sets, σ-Additivity, and Permutable Series

1. Fill in the missing details in the proofs of this section.

1′. Prove Note 3.

2. Show that every open set $A \neq \emptyset$ in E^n is a countable union of disjoint half-open *cubes*.

[Outline: For each natural m, show that E^n is split into such cubes of edge length 2^{-m} by the hyperplanes

$$x_k = \frac{i}{2^m} \quad i = 0, \pm 1, \pm 2, \dots ; \; k = 1, 2, \dots, n,$$

and that the family $\mathcal{C}_m$ of such cubes is countable.

For $m > 1$, let $C_{m1}, C_{m2}, \dots$ be the sequence of those cubes from $\mathcal{C}_m$ (if any) that lie in A but not in any cube C_{sj} with $s < m$.

As A is *open*, $x \in A$ iff $x \in$ some C_{mj}.]

3. Prove that any open set $A \subseteq E^1$ is a countable union of *disjoint* (possibly infinite) open intervals.

[Hint: By Lemma 2, $A = \bigcup_n (a_n, b_n)$. If, say, (a_1, b_1) *overlaps* with some (a_m, b_m), replace both by their union. Continue inductively.]

4. Prove that $\mathcal{C}_\sigma$ is closed under *finite* intersections and *countable* unions.

5. (i) Find $A, B \in \mathcal{C}_\sigma$ such that $A - B \notin \mathcal{C}_\sigma$.

(ii) Show that $\mathcal{C}_\sigma$ is not a semiring.

[Hint: Try $A = E^1$, $B = R$ (the rationals).]

Note. In the following problems, J is countably infinite, $a_i \in E$ (E complete).

6. Prove that

$$\sum_{i\in J} |a_i| < \infty$$

iff for every $\varepsilon > 0$, there is a finite set

$$F \subset J \quad (F \neq \emptyset)$$

such that

$$\sum_{i\in I} |a_i| < \varepsilon$$

for every finite $I \subset J - F$.

[Outline: By Theorem 2, fix $u\colon N \xleftrightarrow{\text{onto}} J$ with

$$\sum_{i\in J} |a_i| = \sum_{n=1}^{\infty} |a_{u_n}|.$$

By Cauchy's criterion,

$$\sum_{n=1}^{\infty} |a_{u_n}| < \infty$$

iff

$$(\forall\, \varepsilon > 0)\ (\exists\, q)\ (\forall\, n > m > q) \quad \sum_{k=m}^{n} |a_{u_k}| < \varepsilon.$$

Let $F = \{u_1, \dots, u_q\}$. If I is as above,

$$(\exists\, n > m > q) \quad \{u_m, \dots, u_n\} \supseteq I;$$

so

$$\sum_{i\in I} |a_i| \le \sum_{k=m}^{n} |a_{u_k}| < \varepsilon.]$$

7. Prove that if

$$\sum_{i\in J} |a_i| < \infty,$$

then for every $\varepsilon > 0$, there is a finite $F \subset J$ $(F \neq \emptyset)$ such that

$$\left|\sum_{i\in J} a_i - \sum_{i\in K} a_i\right| < \varepsilon$$

for each finite $K \supset F$ $(K \subset J)$.
[Hint: Proceed as in Problem 6, with $I = K - F$ and q so large that

$$\left|\sum_{i\in J} a_i - \sum_{i\in F} a_i\right| < \frac{1}{2}\varepsilon \quad \text{and} \quad \left|\sum_{i\in F} a_i\right| < \frac{1}{2}\varepsilon.]$$

8. Show that if

$$J = \bigcup_{n=1}^{\infty} I_n \ (\textit{disjoint}),$$

then

$$\sum_{i\in J} |a_i| = \sum_{n=1}^{\infty} b_n, \text{ where } b_n = \sum_{i\in I_n} |a_i|.$$

(Use Problem 8′ below.)

8′. Show that

$$\sum_{i\in J} |a_i| = \sup_F \sum_{i\in F} |a_i|$$

over all *finite* sets $F \subset J$ $(F \neq \emptyset)$.

[Hint: Argue as in Theorem 2.]

9. Show that if $\emptyset \neq I \subseteq J$, then

$$\sum_{i \in I} |a_i| \leq \sum_{i \in J} |a_i|.$$

[Hint: Use Problem 8′ and Corollary 2 of Chapter 2, §§8–9.]

10. Continuing Problem 8, prove that if

$$\sum_{i \in J} |a_i| = \sum_{n=1}^{\infty} b_n < \infty,$$

then

$$\sum_{i \in J} a_i = \sum_{n=1}^{\infty} c_n \text{ with } c_n = \sum_{i \in I_n} a_i.$$

[Outline: By Problem 9,

$$(\forall\, n) \quad \sum_{i \in I_n} |a_i| < \infty;$$

so

$$c_n = \sum_{i \in I_n} a_i$$

and

$$\sum_{n=1}^{\infty} c_n$$

converge absolutely.

Fix ε and F as in Problem 7. Choose the *largest* $q \in N$ with

$$F \cap I_q \neq \emptyset$$

(why does it exist?), and fix any $n > q$. By Problem 7, $(\forall\, k \leq n)$

$$(\forall\, k \leq n)\ (\exists \text{ finite } F_k \mid J \supseteq F_k \supseteq F \cap I_q)$$
$$(\forall \text{ finite } H_k \mid I_k \supseteq H_k \supseteq F_k) \quad \left| \sum_{i \in H_k} a_i - \sum_{k=1}^{n} c_k \right| < \frac{1}{2}\varepsilon.$$

(Explain!) Let

$$K = \bigcup_{k=1}^{n} H_k;$$

so

$$\left| \sum_{k=1}^{n} c_k - \sum_{i \in J} a_i \right| < \varepsilon$$

and $K \supset F$. By Problem 7,

$$\left| \sum_{i \in K} a_i - \sum_{i \in J} a_i \right| < \varepsilon.$$

Deduce

$$\left|\sum_{k=1}^{n} c_k - \sum_{i\in J} a_i\right| < 2\varepsilon.$$

Let $n \to \infty$; then $\varepsilon \to 0$.]

11. (Double series.) Prove that if one of the expressions

$$\sum_{n,k=1}^{\infty} |a_{nk}|, \quad \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} |a_{nk}|\right), \quad \sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} |a_{nk}|\right)$$

is finite, so are the other two, and

$$\sum_{n,k} a_{nk} = \sum_{n}\left(\sum_{k} a_{nk}\right) = \sum_{k}\left(\sum_{n} a_{nk}\right),$$

with all series involved absolutely convergent.
[Hint: Use Problems 8 and 10, with $J = N \times N$,

$$I_n = \{(n,k) \in J \mid k = 1, 2, \ldots\} \text{ for each } n;$$

so

$$b_n = \sum_{k=1}^{\infty} |a_{nk}| \text{ and } c_n = \sum_{k=1}^{\infty} a_{nk}.$$

Thus obtain

$$\sum_{n,k} a_{nk} = \sum_{n}\sum_{k} a_{nk}.$$

Similarly,

$$\sum_{n,k} a_{nk} = \sum_{k}\sum_{n} a_{n,k}.]$$

§3. More on Set Families[1]

Lebesgue extended his theory far beyond $\mathcal{C}_\sigma$-sets. For a deeper insight, we shall consider set families in more detail, starting with *set rings*. First, we rephrase and supplement our former definition of that notion, given in §1.

Definition 1.

A family $\mathcal{M}$ of subsets of a set S is a *ring* or *set ring* (in S) iff

(i) $\emptyset \in \mathcal{M}$, i.e., the empty set is a member; and

(ii) $\mathcal{M}$ is *closed under finite unions and differences*:

$$(\forall X, Y \in \mathcal{M}) \quad X \cup Y \in \mathcal{M} \text{ and } X - Y \in \mathcal{M}.$$

---

[1] For a *limited approach* (see the preface), this topic may be omitted.

(For *intersections*, see Theorem 1 below.)

If $\mathcal{M}$ is also closed under *countable* unions, we call it a *σ-ring* (in S). Then

$$\bigcup_{i=1}^{\infty} X_i \in \mathcal{M}$$

whenever

$$X_i \in \mathcal{M} \text{ for } i = 1, 2, \ldots .$$

If S itself is a member of a ring (σ-ring) $\mathcal{M}$, we call $\mathcal{M}$ a *set field* (*σ-field*), or a *set algebra* (*σ-algebra*), in S.

Note that S is only a *member* of $\mathcal{M}$, $S \in \mathcal{M}$, not to be confused with $\mathcal{M}$ itself.

The family of *all* subsets of S (the so-called *power set* of S) is denoted by 2^S or $\mathcal{P}(S)$.

Examples.

(a) In any set S, 2^S is a σ-field. (Why?)

(b) The family $\{\emptyset\}$, consisting of $\emptyset$ alone, is a σ-ring; $\{\emptyset, S\}$ is a σ-field in S. (Why?)

(c) The family of all finite (countable) subsets of S is a ring (σ-ring) in S.

(d) For any semiring $\mathcal{C}$, $\mathcal{C}'_s$ is a ring (Theorem 2 in §1). Not so for $\mathcal{C}_\sigma$ (Problem 5 in §2).

Theorem 1. *Any set ring is closed under finite intersections.*

A σ-ring is closed under countable *intersections.*

Proof. Let $\mathcal{M}$ be a σ-ring (the proof for *rings* is similar).

Given a sequence $\{A_n\} \subseteq \mathcal{M}$, we must show that $\bigcap_n A_n \in \mathcal{M}$.

Let

$$U = \bigcup_n A_n.$$

By Definition 1,

$$U \in \mathcal{M} \text{ and } U - A_n \in \mathcal{M},$$

as $\mathcal{M}$ is closed under these operations. Hence

$$\bigcup_n (U - A_n) \in \mathcal{M}$$

and

$$U - \bigcup_n (U - A_n) \in \mathcal{M},$$

or, by duality,

$$\bigcap_n [U - (U - A_n)] \in \mathcal{M},$$

i.e.,

$$\bigcap_n A_n \in \mathcal{M}. \quad \square$$

Corollary 1. *Any set ring (field, σ-ring, σ-field) is also a semiring.*

Indeed, by Theorem 1 and Definition 1, if $\mathcal{M}$ is a ring, then $\emptyset \in \mathcal{M}$ and

$$(\forall\, A, B \in \mathcal{M}) \quad A \cap B \in \mathcal{M} \text{ and } A - B \in \mathcal{M}.$$

Here we may treat $A - B$ as $(A - B) \cup \emptyset$, a union of two disjoint $\mathcal{M}$-sets. Thus $\mathcal{M}$ has all properties of a semiring.

Similarly for σ-rings, fields, etc.

In §1 we saw that any semiring $\mathcal{C}$ can be enlarged to become a ring, $\mathcal{C}'_s$. More generally, we obtain the following result.

Theorem 2. *For any set family $\mathcal{M}$ in a space S ($\mathcal{M} \subseteq 2^S$), there is a unique "smallest" set ring $\mathcal{R}$ such that*

$$\mathcal{R} \supseteq \mathcal{M}$$

("smallest" in the sense that

$$\mathcal{R} \subseteq \mathcal{R}'$$

for any other ring $\mathcal{R}'$ with $\mathcal{R}' \supseteq \mathcal{M}$).

The $\mathcal{R}$ of Theorem 2 is called the ring *generated* by $\mathcal{M}$. Similarly for σ-rings, fields, and σ-fields in S.

Proof. We give the proof for σ-*fields*; it is similar in the other cases.

There surely are σ-fields in S that contain $\mathcal{M}$; e.g., take 2^S. Let $\{\mathcal{R}_i\}$ be the family of *all possible* σ-fields in S such that $\mathcal{R}_i \supseteq \mathcal{M}$. Let

$$\mathcal{R} = \bigcap_i \mathcal{R}_i.$$

We shall show that this $\mathcal{R}$ is the required "smallest" σ-field containing $\mathcal{M}$.

Indeed, by assumption,

$$\mathcal{M} \subseteq \bigcap_i \mathcal{R}_i = \mathcal{R}.$$

We now verify the σ-field properties for $\mathcal{R}$.

(1) We have that

$$(\forall\, i) \quad \emptyset \in \mathcal{R}_i \text{ and } S \in \mathcal{R}_i$$

(for $\mathcal{R}_i$ is a σ-field, by assumption). Hence

$$\emptyset \in \bigcap_i \mathcal{R}_i = \mathcal{R}.$$

Similarly, $S \in \mathcal{R}$. Thus

$$\emptyset, S \in \mathcal{R}.$$

(2) Suppose

$$X, Y \in \mathcal{R} = \bigcap_i \mathcal{R}_i.$$

Then X, Y are in *every* $\mathcal{R}_i$, and so is $X - Y$. Hence $X - Y$ is in

$$\bigcap_i \mathcal{R}_i = \mathcal{R}.$$

Thus $\mathcal{R}$ is *closed under differences.*

(3) Take any sequence

$$\{A_n\} \subseteq \mathcal{R} = \bigcap_i \mathcal{R}_i.$$

Then all A_n are in each $\mathcal{R}_i$. $\bigcup_n A_n$ is in each $\mathcal{R}_i$; so

$$\bigcup_n A_n \in \mathcal{R}.$$

Thus $\mathcal{R}$ is *closed under countable unions.*

We see that $\mathcal{R}$ is indeed a σ-field in S, with $\mathcal{M} \subseteq \mathcal{R}$. As $\mathcal{R}$ is the intersection of *all* $\mathcal{R}_i$ (i.e., *all* σ-fields $\supseteq \mathcal{M}$), we have

$$(\forall\, i) \quad \mathcal{R} \subseteq \mathcal{R}_i;$$

so $\mathcal{R}$ is the *smallest* of such σ-fields.

It is *unique*; for if $\mathcal{R}'$ is another such σ-field, then

$$\mathcal{R} \subseteq \mathcal{R}' \subseteq \mathcal{R}$$

(as both $\mathcal{R}$ and $\mathcal{R}'$ are "smallest"); so

$$\mathcal{R} = \mathcal{R}'. \quad \square$$

Note 1. This proof also shows that the intersection of *any* family $\{\mathcal{R}_i\}$ of σ-fields is a σ-field. Similarly for σ-rings, fields, and rings.

Corollary 2. *The ring $\mathcal{R}$ generated by a semiring $\mathcal{C}$ coincides with*

$$\mathcal{C}_s = \{\textit{all finite unions of } \mathcal{C}\textit{-sets}\}$$

and with

$$\mathcal{C}_s' = \{\text{disjoint } \textit{finite unions of } \mathcal{C}\textit{-sets}\}.$$

Proof. By Theorem 2 in §1, $\mathcal{C}_s'$ is a ring $\supseteq \mathcal{C}$; and

$$\mathcal{C}_s' \subseteq \mathcal{C}_s \subseteq \mathcal{R}$$

(for $\mathcal{R}$ is closed under finite unions, being a *ring* $\supseteq \mathcal{C}$).

Moreover, as $\mathcal{R}$ is the *smallest* ring $\supseteq \mathcal{C}$, we have

$$\mathcal{R} \subseteq \mathcal{C}_s' \subseteq \mathcal{C}_s \subseteq \mathcal{R}.$$

Hence

$$\mathcal{R} = \mathcal{C}_s' = \mathcal{C}_s,$$

as claimed. $\square$

It is much harder to characterize the *σ-ring* generated by a semiring. The following characterization proves useful in theory and as an exercise.[2]

Theorem 3. *The σ-ring $\mathcal{R}$ generated by a semiring $\mathcal{C}$ coincides with the smallest set family $\mathcal{D}$ such that*

(i) $\mathcal{D} \supseteq \mathcal{C}$;

(ii) *$\mathcal{D}$ is closed under countable* disjoint *unions*;

(iii) *$J - X \in \mathcal{D}$ whenever $X \in \mathcal{D}$, $J \in \mathcal{C}$, and $X \subseteq J$.*

Proof. We give a proof *outline*, leaving the details to the reader.

(1) The existence of a *smallest* such $\mathcal{D}$ follows as in Theorem 2. Verify!

(2) Writing briefly AB for $A \cap B$ and A' for $-A$, prove that

$$(A - B)C = A - (AC' \cup BC).$$

(3) For each $I \in \mathcal{D}$, set

$$\mathcal{D}_I = \{A \in \mathcal{D} \mid AI \in \mathcal{D},\ A - I \in \mathcal{D}\}.$$

Then prove that if $I \in \mathcal{C}$, the set family $\mathcal{D}_I$ has the properties (i)–(iii) specified in the theorem. (Use the set identity (2) for property (iii).)

Hence by the *minimality* of $\mathcal{D}$, $\mathcal{D} \subseteq \mathcal{D}_I$. Therefore,

$$(\forall\, A \in \mathcal{D})\ (\forall\, I \in \mathcal{C}) \quad AI \in \mathcal{D} \text{ and } A - I \in \mathcal{D}.$$

(4) Using this, show that $\mathcal{D}_I$ satisfies (i)–(iii) for *any* $I \in \mathcal{D}$.

Deduce

$$\mathcal{D} \subseteq \mathcal{D}_I;$$

so *$\mathcal{D}$ is closed under finite intersections and differences.*

Combining with property (ii), show that $\mathcal{D}$ is a *σ-ring* (see Problem 12 below).

---

[2] It may be deferred until Chapter 8, §8, though.

By its minimality, $\mathcal{D}$ is the *smallest* σ-ring $\supseteq \mathcal{C}$ (for any other such σ-ring clearly satisfies (i)–(iii)).

Thus $\mathcal{D} = \mathcal{R}$, as claimed. □

Definition 2.

Given a set family $\mathcal{M}$, we define (following Hausdorff)

(a) $\mathcal{M}_\sigma = \{$all *countable* unions of $\mathcal{M}$-sets$\}$ (cf. $\mathcal{C}_\sigma$ in §2);

(b) $\mathcal{M}_\delta = \{$all *countable* intersections of $\mathcal{M}$-sets$\}$.

We use $\mathcal{M}_s$ and $\mathcal{M}_d$ for similar notions, with "countable" replaced by "finite."

Clearly,

$$\mathcal{M}_\sigma \supseteq \mathcal{M}_s \supseteq \mathcal{M}$$

and

$$\mathcal{M}_\delta \supseteq \mathcal{M}_d \supseteq \mathcal{M}.$$

Why?

Note 2. Observe that $\mathcal{M}$ is closed under finite (countable) unions iff

$$\mathcal{M} = \mathcal{M}_s \ (\mathcal{M} = \mathcal{M}_\sigma).$$

Verify! Interpret $\mathcal{M} = \mathcal{M}_d$ ($\mathcal{M} = \mathcal{M}_\sigma$) similarly.

In conclusion, we generalize Theorem 1 in §1.

Definition 3.

The *product*

$$\mathcal{M} \dot{\times} \mathcal{N}$$

of two set families $\mathcal{M}$ and $\mathcal{N}$ is the family of all sets of the form

$$A \times B,$$

with $A \in \mathcal{M}$ and $B \in \mathcal{N}$.

(The dot in $\dot{\times}$ is to stress that $\mathcal{M} \dot{\times} \mathcal{N}$ is not really a *Cartesian* product.)

Theorem 4. *If $\mathcal{M}$ and $\mathcal{N}$ are semirings, so is $\mathcal{M} \dot{\times} \mathcal{N}$.*

The proof runs along the same lines as that of Theorem 1 in §1, via the set identities

$$(X \times Y) \cap (X' \times Y') = (X \cap X') \times (Y \cap Y')$$

and

$$(X \times Y) - (X' \times Y') = [(X - X') \times Y] \cup [(X \cap X') \times (Y - Y')].$$

The details are left to the reader.

Note 3. As every ring is a semiring (Corollary 1), the product of two rings (fields, σ-rings, σ-fields) is a *semiring*. However, see Problem 6 below.

Problems on Set Families

1. Verify Examples (a), (b), and (c).

1′. Prove Theorem 1 for *rings*.

2. Show that in Definition 1 "$\emptyset \in \mathcal{M}$" may be replaced by "$\mathcal{M} \neq \emptyset$."
[Hint: $\emptyset = A - A$.]

⇒3. Prove that $\mathcal{M}$ is a field (σ-field) iff $\mathcal{M} \neq \emptyset$, $\mathcal{M}$ is closed under finite (countable) unions, and

$$(\forall A \in \mathcal{M}) \quad -A \in \mathcal{M}.$$

[Hint: $A - B = -(-A \cup B)$; $S = -\emptyset$.]

4. Prove Theorem 2 for *set fields*.

***4′.** Does Note 1 apply to semirings?

5. Prove Note 2.

5′. Prove Theorem 3 in detail.

6. Prove Theorem 4 and show that the product $\mathcal{M} \dot{\times} \mathcal{N}$ of two *rings* need not be a *ring*.
[Hint: Let $S = E^1$ and $\mathcal{M} = \mathcal{N} = 2^S$. Take A, B as in Theorem 1 of §1. Verify that $A - B \notin \mathcal{M} \dot{\times} \mathcal{N}$.]

⇒7. Let $\mathcal{R}, \mathcal{R}'$ be the rings (σ-rings, fields, σ-fields) generated by $\mathcal{M}$ and $\mathcal{N}$, respectively. Prove the following.

(i) If $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{R} \subseteq \mathcal{R}'$.

(ii) If $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{R}$, then $\mathcal{R} = \mathcal{R}'$.

(iii) If

$$\mathcal{M} = \{\text{open intervals in } E^n\}$$

and

$$\mathcal{N} = \{\textit{all} \text{ open sets in } E^n\},$$

then $\mathcal{R} = \mathcal{R}'$.

[Hint: Use Lemma 2 in §2 for (iii). Use the minimality of $\mathcal{R}$ and $\mathcal{R}'$.]

8. Is any of the following a semiring, ring, σ-ring, field, or σ-field? Why?

(a) All *infinite* intervals in E^1.

(b) All open sets in a metric space (S, ρ).

(c) All closed sets in (S, ρ).

(d) All "clopen" sets in (S, ρ).

(e) $\{X \in 2^S \mid -X \text{ finite}\}$.

(f) $\{X \in 2^S \mid -X \text{ countable}\}$.

⇒**9.** Prove that for any sequence $\{A_n\}$ in a ring $\mathcal{R}$, there is

(a) an *expanding* sequence $\{B_n\} \subseteq \mathcal{R}$ such that

$$(\forall\, n) \quad B_n \supseteq A_n$$

and

$$\bigcup_n B_n = \bigcup_n A_n; \text{ and}$$

(b) a *contracting* sequence $C_n \subseteq A_n$, with

$$\bigcap_n C_n = \bigcap_n A_n.$$

(The latter holds in semirings, too.)

[Hint: Set $B_n = \bigcup_1^n A_k$, $C_n = \bigcap_1^n A_k$.]

⇒**10.** The *symmetric difference*, $A \triangle B$, of two sets is defined

$$A \triangle B = (A - B) \cup (B - A).$$

Inductively, we also set

$$\mathop{\triangle}_{k=1}^{1} A_k = A_1$$

and

$$\mathop{\triangle}_{k=1}^{n+1} A_k = \left(\mathop{\triangle}_{k=1}^{n} A_k \right) \triangle A_{n+1}.$$

Show that symmetric differences

(i) are commutative,

(ii) are associative, and

(iii) satisfy the distributive law:

$$(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C).$$

[Hint for (ii): Set $A' = -A$, $A - B = A \cap B'$. Expand $(A \triangle B) \triangle C$ into an expression *symmetric* with respect to A, B, and C.]

11. Prove that $\mathcal{M}$ is a ring iff

(i) $\emptyset \in \mathcal{M}$;

(ii) $(\forall\, A, B \in \mathcal{M})$ $A \triangle B \in \mathcal{M}$ and $A \cap B \in \mathcal{M}$ (see Problem 10); equivalently,

(ii′) $A \triangle B \in \mathcal{M}$ and $A \cup B \in \mathcal{M}$.

[Hint: Verify that

$$A \cup B = (A \triangle B) \triangle (A \cap B)$$

and

$$A - B = (A \cup B) \triangle B,$$

while

$$A \cap B = (A \cup B) \triangle (A \triangle B).]$$

12. Show that a set family $\mathcal{M} \neq \emptyset$ is a σ-ring iff one of the following conditions holds.

(a) $\mathcal{M}$ is closed under countable unions and *proper* differences ($X - Y$ with $X \supseteq Y$);

(b) $\mathcal{M}$ is closed under countable *disjoint* unions, proper differences, and finite intersections; or

(c) $\mathcal{M}$ is closed under countable unions and symmetric differences (see Problem 10).

[Hints: (a) $X - Y = (X \cup Y) - Y$, a *proper* difference.

(b) $X - Y = X - (X \cap Y)$ reduces *any* difference to a *proper* one; then

$$X \cup Y = (X - Y) \cup (Y - X) \cup (X \cap Y)$$

shows that $\mathcal{M}$ is closed under *all* finite unions; so $\mathcal{M}$ is a *ring*. Now use Corollary 1 in §1 for *countable* unions.

(c) Use Problem 11.]

13. From Problem 10, treating $\triangle$ as addition and $\cap$ as multiplication, show that any set ring $\mathcal{M}$ is an *algebraic ring with unity*, i.e., satisfies the six field axioms (Chapter 2, §§1–4), except V(b) (existence of multiplicative inverses).

14. A set family $\mathcal{H}$ is said to be *hereditary* iff

$$(\forall X \in \mathcal{H})\ (\forall Y \subseteq X) \quad Y \in H.$$

Prove the following.

(a) For every family $\mathcal{M} \subseteq 2^S$, there is a "smallest" hereditary ring $\mathcal{H} \supseteq \mathcal{M}$ ($\mathcal{H}$ is said to be *generated* by $\mathcal{M}$). Similarly for σ-rings, fields, and σ-fields.

(b) The hereditary σ-ring generated by $\mathcal{M}$ consists of those sets which can be covered by countably many $\mathcal{M}$-sets.

15. Prove that the field (σ-field) in S, generated by a ring (σ-ring) $\mathcal{R}$, consists exactly of all $\mathcal{R}$-sets and their complements in S.

16. Show that the ring $\mathcal{R}$ generated by a set family $\mathcal{C} \neq \emptyset$ consists of all sets of the form

$$\mathop{\triangle}_{k=1}^{n} A_k$$

(see Problem 10), where each $A_k \in \mathcal{C}_d$ (finite intersection of $\mathcal{C}$-sets).

[Outline: By Problem 11, $\mathcal{R}$ *must* contain the family (call it $\mathcal{M}$) of all such $\triangle_{k=1}^{n} A_k$. (Why?) It remains to show that $\mathcal{M}$ is a ring $\supseteq \mathcal{C}$.

Write $A + B$ for $A \triangle B$ and AB for $A \cap B$; so each $\mathcal{M}$-set is a "sum" of finitely many "products"

$$A_1 A_2 \cdots A_n.$$

By algebra, the "sum" and "product" of two such "polynomials" is such a polynomial itself. Thus

$$(\forall X, Y \in \mathcal{M}) \quad X \triangle Y \text{ and } X \cap Y \in \mathcal{M}.$$

Now use Problem 11.]

17. Use Problem 16 to obtain a new proof of Theorem 2 in §1 and Corollary 2 in the present section.

[Hints: For semirings, $\mathcal{C} = \mathcal{C}_d$. (Why?) Thus in Problem 16, $A_k \in \mathcal{C}$.

Also,

$$(\forall A, B \in \mathcal{C}) \quad A \triangle B = (A - B) \cup (B - A)$$

where $A - B$ and $B - A$ are finite disjoint unions of $\mathcal{C}$-sets. (Why?)

Deduce that $A \triangle B \in \mathcal{C}_s'$ and, by induction,

$$\mathop{\triangle}\limits_{k=1}^{n} A_k \in \mathcal{C}_s';$$

so $\mathcal{R} \subseteq \mathcal{C}_s' \subseteq \mathcal{R}$. (Why?)]

18. Given a set A and a set family $\mathcal{M}$, let

$$A \Cap \mathcal{M}$$

be the family of all sets $A \cap X$, with $X \in \mathcal{M}$; similarly,

$$\mathcal{N} \Cup (\mathcal{M} \mathbin{\dot{-}} A) = \{\text{all sets } Y \cup (X - A), \text{ with } Y \in \mathcal{N},\ X \in \mathcal{M}\}, \text{ etc.}$$

Show that if $\mathcal{M}$ generates the ring $\mathcal{R}$, then $A \Cap \mathcal{M}$ generates the ring

$$\mathcal{R}' = A \Cap \mathcal{R}.$$

Similarly for σ-rings, fields, σ-fields.

[Hint for *rings*: Prove the following.

(i) $A \Cap \mathcal{R}$ is a ring.

(ii) $\mathcal{M} \subseteq \mathcal{R}' \Cup (\mathcal{R} \mathbin{\dot{-}} A)$, with $\mathcal{R}'$ as above.

(iii) $\mathcal{R} \Cup (\mathcal{R} \mathbin{\dot{-}} A)$ is a ring (call it $\mathcal{N}$).

(iv) By (ii), $\mathcal{R} \subseteq \mathcal{N}$, so $A \Cap \mathcal{R} \subseteq A \Cap \mathcal{N} \subseteq \mathcal{R}'$.

(v) $A \Cap \mathcal{R} \supseteq \mathcal{R}'$ (for $A \Cap \mathcal{R} \supseteq A \Cap \mathcal{M}$).

Hence $\mathcal{R}' = A \Cap \mathcal{R}$.]

§4. Set Functions. Additivity. Continuity

I. The letter "v" in vA may be treated as a certain *function symbol* that assigns a numerical value (called "volume") to the set A. So far we have defined such "volumes" for all intervals, then for $\mathcal{C}$-simple sets, and even for $\mathcal{C}_\sigma$-sets in E^n.

Mathematically this means that the volume function v has been defined first on $\mathcal{C}$ (the intervals), then on $\mathcal{C}_s'$ ($\mathcal{C}$-simple sets), and finally on $\mathcal{C}_\sigma$.

Thus we have a function v which assigns values ("volumes") not just to single points, as ordinary "point functions" do, but to whole sets, each set being treated as *one* thing.

In other words, the domain of the function v is not just a set of points, but a *set family* ($\mathcal{C}$, $\mathcal{C}_s'$, or $\mathcal{C}_\sigma$).

The "volumes" assigned to such sets are the *function values* (for $\mathcal{C}$ and $\mathcal{C}_s'$-sets they are real numbers; for $\mathcal{C}_\sigma$-sets they may reach $+\infty$). This is symbolized by

$$v\colon \mathcal{C} \to E^1$$

or

$$v\colon \mathcal{C}_\sigma \to E^*;$$

more precisely,

$$v\colon \mathcal{C}_\sigma \to [0, \infty],$$

since volume is *nonnegative*.

It is natural to call v a *set function* (as opposed to ordinary *point functions*). As we shall see, there are many other set functions. The function *values* need not be real; they may be complex numbers or vectors. This agrees with our general definition of a function as a certain *set of ordered pairs* (Definition 3 in Chapter 1, §§4–7); e.g.,

$$v = \begin{pmatrix} A & B & C & \cdots \\ vA & vB & vC & \cdots \end{pmatrix}.$$

Here the domain consists of certain *sets* $A, B, C, \ldots$. This leads us to the following definition.

Definition 1.

A *set function* is a mapping

$$s\colon \mathcal{M} \to E$$

whose domain is a *set family* $\mathcal{M}$.

The range space E is assumed to be E^1, E^*, C (the complex field), E^n, or another normed space. Thus s may be real, extended real, complex, or vector valued.

To each set $X \in \mathcal{M}$, the function s assigns a unique function value denoted $s(X)$ or sX (which is an element of the range space E).

We say that s is *finite* on a set family $\mathcal{N} \subseteq \mathcal{M}$ iff

$$(\forall\, X \in \mathcal{N}) \quad |sX| < \infty;$$

briefly, $|s| < \infty$ *on* $\mathcal{N}$. (This is automatic if s is complex or vector valued.)

We call s *semifinite* if at least one of $\pm\infty$ is excluded as function value, e.g., if $s \geq 0$ on $\mathcal{M}$; i.e.,

$$s\colon \mathcal{M} \to [0, \infty].$$

(The symbol ∞ stands for $+\infty$ throughout.)

Definition 2.

A set function

$$s\colon \mathcal{M} \to E$$

is called *additive* (or *finitely additive*) on $\mathcal{N} \subseteq \mathcal{M}$ iff for any finite *disjoint* union $\bigcup_k A_k$, we have

$$\sum_k sA_k = s\Big(\bigcup_k A_k\Big),$$

provided $\bigcup_k A_k$ and all the A_k are $\mathcal{N}$-sets.

If this also holds for *countable* disjoint unions, s is called *σ-additive* (or *countably additive* or *completely additive*) on $\mathcal{N}$.

If $\mathcal{N} = \mathcal{M}$ here, we simply say that s is *additive* (*σ-additive*, respectively).

Note 1. As $\bigcup A_k$ is independent of the order of the A_k, σ-additivity presupposes and implies that the series

$$\sum sA_k$$

is *permutable* (§2) for any disjoint sequence

$$\{A_k\} \subseteq \mathcal{N}.$$

(The *partial* sums do exist, by our conventions (2^*) in Chapter 4, §4.)

The set functions in the examples below are additive; v is even σ-additive (Corollary 1 in §2).

Examples (b)–(d) show that set functions may arise from ordinary "*point* functions."

Examples.

(a) The volume function $v\colon \mathcal{C} \to E^1$ on $\mathcal{C}$ (= intervals in E^n), discussed above, is called the *Lebesgue premeasure* (in E^n).

(b) Let $\mathcal{M} = \{\text{all finite intervals } I \subset E^1\}$.

Given $f\colon E^1 \to E$, set

$$(\forall\, I \in \mathcal{M}) \quad sI = V_f[\overline{I}],$$

the total variation of f on the *closure* of I (Chapter 5, §7).

Then $s\colon \mathcal{M} \to [0, \infty]$ is additive by Theorem 1 of Chapter 5, §7.

(c) Let $\mathcal{M}$ and f be as in Example (b).

Suppose f has an *antiderivative* (Chapter 5, §5) on E^1. For each interval X with endpoints $a, b \in E^1$ $(a \leq b)$, set

$$sX = \int_a^b f.$$

This yields a set function $s\colon \mathcal{M} \to E$ (real, complex, or vector valued), additive by Corollary 6 in Chapter 5, §5.

(d) Let $\mathcal{C} = \{$all finite intervals in $E^1\}$.

Suppose

$$\alpha : E^1 \to E^1$$

has finite one-sided limits

$$\alpha(p+) \text{ and } \alpha(p-)$$

at each $p \in E^1$. The *Lebesgue–Stieltjes* (*LS*) *function*

$$s_\alpha\colon \mathcal{C} \to E^1$$

(important for *Lebesgue–Stieltjes integration*) is defined as follows.

Set $s_\alpha \emptyset = 0$. For *nonvoid* intervals, including $[a, a] = \{a\}$, set

$$\begin{aligned} s_\alpha[a, b] &= \alpha(b+) - \alpha(a-), \\ s_\alpha(a, b] &= \alpha(b+) - \alpha(a+), \\ s_\alpha[a, b) &= \alpha(b-) - \alpha(a-), \text{ and} \\ s_\alpha(a, b) &= \alpha(b-) - \alpha(a+). \end{aligned}$$

For the properties of s_α see Problem 7 ff., below.

(e) Let mX be the mass concentrated in the part X of the physical space S. Then m is a *nonnegative* set function defined on

$$2^S = \{\text{all subsets } X \subseteq S\} \ (\S 3).$$

If instead mX were the electric load of X, then m would be *sign changing*.

II. The rest of this section is redundant for a "limited approach."

Lemmas. *Let $s\colon \mathcal{M} \to E$ be additive on $\mathcal{N} \subseteq \mathcal{M}$. Let*

$$A, B \in \mathcal{N}, \ A \subseteq B.$$

Then we have the following.

(1) *If $|sA| < \infty$ and $B - A \in \mathcal{N}$, then*

$$s(B - A) = sB - sA \ (\text{"}subtractivity\text{"}).$$

(2) *If* $\emptyset \in \mathcal{N}$, *then* $s\emptyset = 0$ *provided* $|sX| < \infty$ *for at least one* $X \in \mathcal{N}$.

(3) *If* $\mathcal{N}$ *is a semiring, then* $sA = \pm\infty$ *implies* $|sB| = \infty$. *Hence*

$$|sB| < \infty \Rightarrow |sA| < \infty.$$

If further s *is semifinite then*

$$sA = \pm\infty \Rightarrow sB = \pm\infty$$

(*same sign*).

Proof.

(1) As $B \supseteq A$, we have

$$B = (B - A) \cup A \ (disjoint);$$

so by additivity,

$$sB = s(B - A) + sA.$$

If $|sA| < \infty$, we may transpose to get

$$sB - sA = s(B - A),$$

as claimed.

(2) Hence

$$s\emptyset = s(X - X) = sX - sX = 0$$

if $X, \emptyset \in \mathcal{N}$, and $|sX| < \infty$.

(3) If $\mathcal{N}$ is a semiring, then

$$B - A = \bigcup_{k=1}^{n} A_k \ (disjoint)$$

for some $\mathcal{N}$-sets A_k; so

$$B = A \cup \bigcup_{k=1}^{n} A_k \ (disjoint).$$

By additivity,

$$sB = sA + \sum_{k=1}^{n} sA_k;$$

so by our conventions,

$$|sA| = \infty \Rightarrow |sB| = \infty.$$

If, further, s is semifinite, one of $\pm\infty$ is excluded. Thus sA and sB, if infinite, must have *the same* sign. This completes the proof. □

In §§1 and 2, we showed how to extend the notion of volume from intervals to a larger set family, *preserving additivity*. We now generalize this idea.

Theorem 1. *If*

$$s\colon \mathcal{C} \to E$$

is additive on $\mathcal{C}$, *an* arbitrary semiring, *there is a unique set function*

$$\bar{s}\colon \mathcal{C}_s \to E,$$

additive on $\mathcal{C}_s$, *with* $\bar{s} = s$ *on* $\mathcal{C}$, *i.e.,*

$$\bar{s}X = sX \text{ for } X \in \mathcal{C}.$$

We call $\bar{s}$ the *additive extension* of s to $\mathcal{C}_s = \mathcal{C}'_s$ (Corollary 2 in §3).

Proof. If $s \geq 0$ ($s\colon \mathcal{C} \to [0, \infty]$), proceed as in Lemma 1 and Corollary 2, all of §1.

The *general* proof (which may be omitted or deferred) is as follows.

Each $X \in \mathcal{C}'_s$ has the form

$$X = \bigcup_{i=1}^{m} X_i \text{ (disjoint)}, \quad X_i \in \mathcal{C}.$$

Thus if $\bar{s}$ is to be additive, the *only* way to define it is to set

$$\bar{s}X = \sum_{i=1}^{m} sX_i.$$

This already makes $\bar{s}$ *unique*, provided we show that

$$\sum_{i=1}^{m} sX_i$$

does not depend on the particular decomposition

$$X = \bigcup_{i=1}^{m} X_i$$

(otherwise, all is ambiguous).

Then take any other decomposition

$$X = \bigcup_{k=1}^{n} Y_k \text{ (disjoint)}, \quad Y_k \in \mathcal{C}.$$

Additivity implies

$$(\forall\, i, k) \quad sX_i = \sum_{k=1}^{n} s(X_i \cap Y_k) \text{ and } sY_k = \sum_{i=1}^{m} s(X_i \cap Y_k).$$

(Verify!) Hence

$$\sum_{i=1}^{m} sX_i = \sum_{i,k} s(X_i \cap Y_k) = \sum_{k=1}^{n} sY_k.$$

Thus, indeed, it does not matter which particular decomposition we choose, and our definition of $\bar{s}$ is unambiguous.

If $X \in \mathcal{C}$, we may choose (say)

$$X = \bigcup_{i=1}^{1} X_i, \ X_1 = X;$$

so

$$\bar{s}X = sX_1 = sX;$$

i.e., $\bar{s} = s$ on $\mathcal{C}$, as required.

Finally, for the additivity of $\bar{s}$, let

$$A = \bigcup_{k=1}^{m} B_k \ (\textit{disjoint}), \quad A, B_k \in \mathcal{C}_s'.$$

Here we may set

$$B_k = \bigcup_{i=1}^{n_k} C_{ki} \ (\textit{disjoint}), \quad C_{ki} \in \mathcal{C}.$$

Then

$$A = \bigcup_{k,i} C_{ki} \ (\textit{disjoint});$$

so by our definition of $\bar{s}$,

$$\bar{s}A = \sum_{k,i} sC_{ki} = \sum_{k=1}^{m} \left(\sum_{i=1}^{n_k} sC_{ki} \right) = \sum_{k=1}^{m} \bar{s}B_k,$$

as required. $\square$

Continuity. We write $X_n \nearrow X$ to mean that

$$X = \bigcup_{n=1}^{\infty} X_n$$

and $\{X_n\}\uparrow$, i.e.,

$$X_n \subseteq X_{n+1}, \quad n = 1, 2, \ldots.$$

Similarly, $X_n \searrow X$ iff

$$X = \bigcap_{n=1}^{\infty} X_n$$

and $\{X_n\}\downarrow$, i.e.,

$$X_n \supseteq X_{n+1}, \quad n = 1, 2, \ldots .$$

In both cases, we set

$$X = \lim_{n\to\infty} X_n.$$

This suggests the following definition.

Definition 3.

A set function $s\colon \mathcal{M} \to E$ is said to be

(i) *left continuous* (on $\mathcal{M}$) iff

$$sX = \lim_{n\to\infty} sX_n$$

whenever $X_n \nearrow X$ and $X, X_n \in \mathcal{M}$;

(ii) *right continuous* iff

$$sX = \lim_{n\to\infty} sX_n$$

whenever $X_n \searrow X$, with $X, X_n \in \mathcal{M}$ and $|sX_j| < \infty$.

Thus in case (i),

$$\lim_{n\to\infty} sX_n = s \bigcup_{n=1}^{\infty} X_n$$

if all X_n and $\bigcup_{n=1}^{\infty} X_n$ are $\mathcal{M}$-sets.

In case (ii),

$$\lim_{n\to\infty} sX_n = s \bigcap_{n=1}^{\infty} X_n$$

if all X_n and $\bigcap_{n=1}^{\infty} X_n$ are in $\mathcal{M}$, and $|sX_1| < \infty$.

Note 2. The last restriction applies to *right* continuity only. (We choose simply to exclude from consideration sequences $\{X_n\}\downarrow$, with $|sX_1| = \infty$; see Problem 4.)

Theorem 2. *If $s\colon \mathcal{C} \to E$ is σ-additive and semifinite on $\mathcal{C}$, a* semiring, *then s is both left and right continuous (briefly, continuous).*

Proof. We sketch the proof for *rings*; for *semirings*, see Problem 1.

Left continuity. Let $X_n \nearrow X$ with $X_n, X \in \mathcal{C}$ and

$$X = \bigcup_{n=1}^{\infty} X_n.$$

If $sX_n = \pm\infty$ for some n, then (Lemma 3)

$$sX = sX_m = \pm\infty \text{ for } m \geq n,$$

since $X \supseteq X_m \supseteq X_n$; so

$$\lim sX_m = \pm\infty = sX,$$

as claimed.

Thus assume *all* sX_n finite; so $s\emptyset = 0$, by Lemma 2.

Set $X_0 = \emptyset$. As is easily seen,

$$X = \bigcup_{n=1}^{\infty} X_n = \bigcup_{n=1}^{\infty} (X_n - X_{n-1}) \ (\textit{disjoint}),$$

and

$$(\forall\, n) \quad X_n - X_{n-1} \in \mathcal{C} \ (\text{a } \textit{ring}).$$

Also,

$$(\forall\, m \geq n) \quad X_m = \bigcup_{n=1}^{m} (X_n - X_{n-1}) \ (\textit{disjoint}).$$

(Verify!) Thus by additivity,

$$sX_m = \sum_{n=1}^{m} s(X_n - X_{n-1}),$$

and by the assumed σ-additivity,

$$\begin{aligned} sX = s\bigcup_{n=1}^{\infty}(X_n - X_{n-1}) &= \sum_{n=1}^{\infty} s(X_n - X_{n-1}) \\ &= \lim_{m\to\infty} \sum_{n=1}^{m} s(X_n - X_{n-1}) = \lim_{m\to\infty} sX_m, \end{aligned}$$

as claimed.

Right continuity. Let $X_n \searrow X$ with $X, X_n \in \mathcal{C}$,

$$X = \bigcap_{n=1}^{\infty} X_n,$$

and

$$|sX_1| < \infty.$$

As $X \subseteq X_n \subseteq X_1$, Lemma 3 yields that

$$(\forall\, n) \quad |sX_n| < \infty$$

and $|sX| < \infty$.

As

$$X = \bigcap_{k=1}^{\infty} X_k,$$

we have

$$(\forall\, n) \quad X_n = X \cup \bigcup_{k=n+1}^{\infty} (X_{k-1} - X_k)\ (\textit{disjoint}).$$

(Verify!) Thus by σ-additivity,

$$(\forall\, n) \quad sX_n = sX + \sum_{k=n+1}^{\infty} s(X_{k-1} - X_k),$$

with $|sX| < \infty$, $|sX_n| < \infty$ (see above).

Hence the sum

$$\sum_{k=n+1}^{\infty} s(X_{k-1} - X_k) = sX_n - sX$$

is *finite*. Therefore, it tends to 0 as $n \to \infty$ (being the "remainder term" of a *convergent* series). Thus $n \to \infty$ yields

$$\lim_{n\to\infty} sX_n = sX + \lim \sum_{k=n+1}^{\infty} s(X_{k-1} - X_k) = sX,$$

as claimed. □

Problems on Set Functions

1. Prove Theorem 2 in detail for *semirings*.

[Hint: We know that

$$X_n - X_{n-1} = \bigcup_{i=1}^{m_n} Y_{ni}\ (\textit{disjoint})$$

for some $Y_{ni} \in \mathcal{C}$, so

$$\bar{s}(X_n - X_{n-1}) = \sum_{i=1}^{m_n} sY_{ni},$$

with $\bar{s}$ as in Theorem 1.]

2. Let s be additive on $\mathcal{M}$, a *ring*. Prove that s is also σ-additive provided s is either

(i) left continuous, or

(ii) finite on $\mathcal{M}$ and right-continuous *at* $\emptyset$; i.e.,

$$\lim_{n\to\infty} sX_n = 0$$

when $X_n \searrow \emptyset$ ($X_n \in \mathcal{M}$).

[Hint: Let

$$A = \bigcup_n A_n\ (\textit{disjoint}), \quad A, A_n \in \mathcal{M}.$$

Set

$$X_n = \bigcup_{k=1}^{n} A_k,\ Y_n = A - X_n.$$

Verify that $X_n, Y_n \in \mathcal{M}$, $X_n \nearrow A$, $Y_n \searrow \emptyset$.

In case (i),

$$sA = \lim sX_n = \sum_{k=1}^{\infty} sA_k.$$

(Why?)

For (ii), use the Y_n.]

3. Let

$$\mathcal{M} = \{\text{all intervals in the } \textit{rational field } R \subset E^1\}.$$

Let

$$sX = b - a$$

if a, b are the endpoints of $X \in \mathcal{M}$ $(a, b \in R,\ a \leq b)$. Prove that

(i) $\mathcal{M}$ is a semiring;

(ii) s is continuous;

(iii) s is additive but *not* σ-additive; thus Problem 2 fails for *semirings*.

[Hint: R is *countable*. Thus each $X \in \mathcal{M}$ is a countable union of singletons $\{x\} = [x, x]$; hence $sX = 0$ if s were σ-additive.]

3′. Let $N = \{\text{naturals}\}$. Let

$$\mathcal{M} = \{\text{all finite subsets of } N \text{ and their complements in } N\}.$$

If $X \in \mathcal{M}$, let $sX = 0$ if X is finite, and $sX = \infty$ otherwise. Show that

(i) $\mathcal{M}$ is a set field;

(ii) s is right continuous and additive, but *not* σ-additive.

Thus Problem 2(ii) fails if s is not finite.

4. Let

$$\mathcal{C} = \{\text{finite and infinite intervals in } E^1\}.$$

If a, b are the endpoints of an interval X $(a, b \in E^*,\ a < b)$, set

$$sX = \begin{cases} b - a, & a < b, \\ 0, & a = b. \end{cases}$$

Show that s is σ-additive on $\mathcal{C}$, a *semiring*.

Let

$$X_n = (n, \infty);$$

so $sX_n = \infty - n = \infty$ and $X_n \searrow \emptyset$. (Verify!) Yet

$$\lim sX_n = \infty \neq s\emptyset.$$

Does this contradict Theorem 2?

5. Fill in the missing proof details in Theorem 1.

6. Let s be additive on $\mathcal{M}$. Prove the following.

(i) If $\mathcal{M}$ is a ring or semiring, so is

$$\mathcal{N} = \{X \in \mathcal{M} \mid |sX| < \infty\}$$

if $\mathcal{N} \neq \emptyset$.

(ii) If $\mathcal{M}$ is generated by a set family $\mathcal{C}$, with $|s| < \infty$ on $\mathcal{C}$, then $|s| < \infty$ on $\mathcal{M}$.

[Hint: Use Problem 16 in §3.]

⇒**7.** (Lebesgue–Stieltjes set functions.) Let α and s_α be as in Example (d). Prove the following.

(i) $s_\alpha \geq 0$ on $\mathcal{C}$ iff $\alpha\uparrow$ on E^1 (see Theorem 2 in Chapter 4, §5).

(ii) $s_\alpha\{p\} = s_\alpha[p, p] = 0$ iff α is continuous at p.

(iii) s_α is *additive.*

[Hint: If

$$A = \bigcup_{i=1}^{n} A_i \ (\textit{disjoint}),$$

the intervals A_{i-1}, A_i must be *adjacent*. For *two* such intervals, consider all cases like

$$(a, b] \cup (b, c),\ [a, b) \cup [b, c], \text{ etc.}$$

Then use induction on n.]

(iv) If α is *right* continuous at a and b, then

$$s_\alpha(a, b] = \alpha(b) - \alpha(b).$$

If α is continuous at a and b, then

$$s_\alpha[a, b] = s_\alpha(a, b] = s_\alpha[a, b) = s_\alpha(a, b).$$

(v) If $\alpha\uparrow$ on E^1, then s_α satisfies Lemma 1 and Corollary 2 in §1 (same proof), as well as Lemma 1, Theorem 1, Corollaries 1–4, and Note 3 in §2 (everything except Corollaries 5 and 6).

[Hint: Use (i) and (iii). For Lemma 1 in §2, take first a *half-open* $B = (a, b]$; use the definition of a *right-side* limit along with Theorems 1 and 2 in Chapter 4, §5, to prove

$$(\forall \varepsilon > 0)\ (\exists c > b) \quad 0 \leq \alpha(c-) - \alpha(b+) < \varepsilon;$$

then set $C = (a, c)$. Similarly for $B = [a, b)$, etc. and for the *closed* interval $A \subseteq B$.]

(vi) If $\alpha(x) = x$ then $s_\alpha = v$, the volume (or *length*) function in E^1.

8. Construct LS set functions (Example (d)), with $\alpha\!\uparrow$ (see Problem 7(v)), so that

(i) $s_\alpha[0,1] \neq s_\alpha[1,2]$;

(ii) $s_\alpha E^1 = 1$ (after extending s_α to $\mathcal{C}_\sigma$-sets in E^1);

(ii′) $s_\alpha E^1 = c$ for a fixed $c \in (0,\infty)$;

(iii) $s_\alpha\{0\} = 1$ and $s_\alpha[0,1] > s_\alpha(0,1]$.

Describe s_α if $\alpha(x) = [x]$ (the integral part of x).
[Hint: See Figure 16 in Chapter 4, §1.]

9. For an *arbitrary* $\alpha\colon E^1 \to E^1$, define $\sigma_\alpha\colon \mathcal{C} \to E^1$ by

$$\sigma_\alpha[a,b] = \sigma_\alpha(a,b] = \sigma_\sigma[a,b) = \sigma_\alpha(a,b) = \alpha(b) - \alpha(a)$$

(the *original* Stieltjes method). Prove that σ_α is additive *but not σ-additive* unless α is continuous (*for Theorem* 2 *fails*).

§5. Nonnegative Set functions. Premeasures. Outer Measures

We now concentrate on *nonnegative* set functions

$$m : \mathcal{M} \to [0,\infty]$$

(we mostly denote them by m or μ). Such functions have the advantage that

$$\sum_{n=1}^{\infty} mX_n$$

exists and is *permutable* (Theorem 2 in §2) for *any* sets $X_n \in \mathcal{M}$, since $mX_n \geq 0$. Several important notions apply to such functions (only). They "mimic" §§1 and 2.

Definition 1.

A set function

$$m : \mathcal{M} \to [0,\infty]$$

is said to be

(i) *monotone* (on $\mathcal{M}$) iff

$$mX \leq mY$$

whenever

$$X \subseteq Y \text{ and } X, Y \in \mathcal{M};$$

(ii) (finitely) *subadditive* (on $\mathcal{M}$) iff for any finite union

$$\bigcup_{k=1}^{n} Y_k,$$

we have

$$mX \leq \sum_{k=1}^{m} mY_k \tag{1}$$

whenever $X, Y_k \in \mathcal{M}$ and

$$X \subseteq \bigcup_{k=1}^{n} Y_k \text{ (disjoint or } \textit{not}\text{);}$$

(iii) *σ-subadditive* (on $\mathcal{M}$) iff (1) holds for countable unions, too.

Recall that $\{Y_k\}$ is called a *covering* of X iff

$$X \subseteq \bigcup_{k} Y_k.$$

We call it an $\mathcal{M}$-covering of X if all Y_k are $\mathcal{M}$-sets. We now obtain the following corollary.

Corollary 1. *Subadditivity implies monotonicity.*

Take $n = 1$ in formula (1).

Corollary 2. *If* $m : \mathcal{C} \to [0, \infty]$ *is additive* (σ*-additive*) *on* $\mathcal{C}$, *a* semiring, *then* m *is also subadditive* (σ*-subadditive*, *respectively*), *hence monotone*, *on* $\mathcal{C}$.

The proof is a mere repetition of the argument used in Lemma 1 in §1.

Taking $n = 1$ in formula (ii) there, we obtain finite subadditivity.

For σ-subadditivity, one only has to use countable unions instead of finite ones.

Note 1. The converse *fails*: subadditivity does not imply additivity.

Note 2. Of course, Corollary 2 applies to *rings*, too (see Corollary 1 in §3).

Definition 2.

A *premeasures* is a set function

$$\mu \colon \mathcal{C} \to [0, \infty]$$

such that

$$\emptyset \in \mathcal{C} \text{ and } \mu\emptyset = 0.$$

($\mathcal{C}$ may, but *need not*, be a semiring.)

A *premeasure space* is a triple

$$(S, \mathcal{C}, \mu),$$

where $\mathcal{C}$ is a family of subsets of S (briefly, $\mathcal{C} \subseteq 2^S$) and

$$\mu \colon \mathcal{C} \to [0, \infty]$$

is a premeasure. In this case, $\mathcal{C}$-sets are also called *basic* sets.

If

$$A \subseteq \bigcup_n B_n,$$

with $B_n \in \mathcal{C}$, the sequence $\{B_n\}$ is called a *basic covering* of A, and

$$\sum_n \mu B_n$$

is a *basic covering value* of A; $\{B_n\}$ may be finite or infinite.

Examples.

(a) The volume function v on $\mathcal{C}$ ($=$ intervals in E^n) is a premeasure, as $v \geq 0$ and $v\emptyset = 0$. $(E^n, \mathcal{C}, v)$ is the *Lebesgue premeasure space.*

(b) The LS set function s_α is a premeasure if $\alpha\!\uparrow$ (see Problem 7 in §4). We call it the *α-induced Lebesgue–Stieltjes (LS) premeasure in E^1.*

We now develop a method for constructing *σ-subadditive* premeasures. (This is a first step toward achieving σ-additivity; see §4.)

Definition 3.

For any premeasure space $(S, \mathcal{C}, \mu)$, we define the *μ-induced outer measure* m^* on 2^S ($=$ all subsets of S) by setting, for each $A \subseteq S$,

$$m^*A = \inf\left\{\sum_n \mu B_n \;\middle|\; A \subseteq \bigcup_n B_n,\ B_n \in \mathcal{C}\right\}, \tag{2}$$

i.e., m^*A (called the *outer measure of A*) is the *glb of all basic covering values of A.*

If $\mu = v$, m^* is called the *Lebesgue outer measure* in E^n.

Note 3. If A has no basic coverings, we set $m^*A = \infty$. More generally, we make the convention that $\inf \emptyset = +\infty$.

Note 4. By the properties of the glb, we have

$$(\forall\, A \subseteq S) \quad 0 \leq m^*A.$$

If $A \in \mathcal{C}$, then $\{A\}$ is a basic covering; so

$$m^*A \leq \mu A.$$

In particular, $m^*\emptyset = \mu\emptyset = 0$.

Theorem 1.[1] *The set function m^* so defined is σ-subadditive on 2^S.*

Proof. Given

$$A \subseteq \bigcup_n A_n \subset S,$$

we must show that

$$m^*A \leq \sum_n m^*A_n.$$

This is trivial if $m^*A_n = \infty$ for some n. Thus assume

$$(\forall\, n) \quad m^*A_n < \infty$$

and fix $\varepsilon > 0$.

By Note 3, each A_n *has* a basic covering

$$\{B_{nk}\}, \quad k = 1, 2, \ldots$$

(otherwise, $m^*A_n = \infty$). By properties of the glb, we can choose the B_{nk} so that

$$(\forall\, n) \quad \sum_k \mu B_{nk} < m^*A_n + \frac{\varepsilon}{2^n}.$$

(Explain from (2)). The sets B_{nk} (for all n and all k) form a countable basic covering of *all* A_n, hence of A. Thus by Definition 3,

$$m^*A \leq \sum_n \left(\sum_k \mu B_{nk} \right) \leq \sum_n \left(m^*A_n + \frac{\varepsilon}{2^n} \right) \leq \sum^n m^*A_n + \varepsilon.$$

As ε is arbitrary, we can let $\varepsilon \to 0$ to obtain the desired result. □

Note 5. In view of Theorem 1, we now generalize the notion of an *outer measure in S* to mean *any* σ-subadditive premeasure defined on all of 2^S.

By Note 4, $m^* \leq \mu$ on $\mathcal{C}$, not $m^* = \mu$ in general. However, we obtain the following result.

Theorem 2. *With m^* as in Definition 3, we have $m^* = \mu$ on $\mathcal{C}$ iff μ is σ-subadditive on $\mathcal{C}$. Hence, in this case, m^* is an* extension *of μ.*

Proof. Suppose μ is σ-subadditive and fix any $A \in \mathcal{C}$. By Note 4,

$$m^*A \leq \mu A.$$

We shall show that

$$\mu A \leq m^*A,$$

---

[1] Theorems 1–3 are redundant for a "limited approach" (see the preface). Pass to Chapter 8, §1.

too, and hence $\mu A = m^* A$.

Now, as $A \in \mathcal{C}$, A surely has basic coverings, e.g., $\{A\}$. Take *any* basic covering:

$$A \subseteq \bigcup_n B_n, \quad B_n \in \mathcal{C}.$$

As μ is σ-subadditive,

$$\mu A \leq \sum_n \mu B_n.$$

Thus μA does not exceed *any* basic covering values of A; so it cannot exceed their glb, $m^* A$. Hence $\mu = m^*$, indeed.

Conversely, if $\mu = m^*$ on $\mathcal{C}$, then the σ-subadditivity of m^* (Theorem 1) implies that of μ (on $\mathcal{C}$). Thus all is proved. $\square$

Note 6. If, in (2), we allow only *finite* basic coverings, then the μ-induced set function is called the *μ-induced outer content*, c^*. It is only *finitely* subadditive, in general.

In particular, if $\mu = v$ (Lebesgue premeasure), we speak of the *Jordan outer content* in E^n. (It is superseded by Lebesgue theory but still occurs in courses on Riemann integration.)

We add two more definitions related to the notion of *coverings*.

Definition 4.

A set function $s\colon \mathcal{M} \to E$ ($\mathcal{M} \subseteq 2^S$) is called *$\sigma$-finite* iff every $X \in \mathcal{M}$ can be covered by a sequence of $\mathcal{M}$-sets X_n, with

$$|sX_n| < \infty \quad (\forall\, n).$$

Any set $A \subseteq S$ which can be so covered is said to be *σ-finite with respect to s* (briefly, (s) *σ-finite*).

If the whole space S can be so covered, we say that s is *totally σ-finite*.

For example, the Lebesgue premeasure v on E^n is totally σ-finite.

Definition 5.

A set function $s\colon \mathcal{M} \to E^*$ is said to be *regular with respect to a set family $\mathcal{A}$* (briefly, *$\mathcal{A}$-regular*) iff for each $A \in \mathcal{M}$,

$$sA = \inf\{sX \mid A \subseteq X,\ X \in \mathcal{A}\}; \tag{3}$$

that is, sA is the glb of all sX, with $A \subseteq X$ and $X \in \mathcal{A}$.

These notions are important for our later work. At present, we prove only one theorem involving Definitions 3 and 5.

Theorem 3. *For any premeasure space* $(S, \mathcal{C}, \mu)$, *the* μ*-induced outer measure* m^* *is* $\mathcal{A}$*-regular whenever*

$$\mathcal{C}_\sigma \subseteq \mathcal{A} \subseteq 2^S.$$

Thus in this case,

$$(\forall\, A \subseteq S) \quad m^*A = \inf\{m^*X \mid A \subseteq X,\ X \in \mathcal{A}\}. \tag{4}$$

Proof. As m^* is monotone, m^*A is surely a lower bound of

$$\{m^*X \mid A \subseteq X,\ X \in \mathcal{A}\}.$$

We must show that there is no *greater* lower bound.

This is trivial if $m^*A = \infty$.

Thus let $m^*A < \infty$; so A *has* basic coverings (Note 3). Now fix any $\varepsilon > 0$. By formula (2), there is a basic covering $\{B_n\} \subseteq \mathcal{C}$ such that

$$A \subseteq \bigcup_n B_n$$

and

$$m^*A + \varepsilon > \sum_n \mu B_n \geq \sum_n m^*B_n \geq m^* \bigcup_n B_n.$$

(m^* is σ-subadditive!)

Let

$$X = \bigcup_n B_n.$$

Then X is in $\mathcal{C}_\sigma$, hence in $\mathcal{A}$, and $A \subseteq X$. Also,

$$m^*A + \varepsilon > m^*X.$$

Thus $m^*A + \varepsilon$ is *not* a lower bound of

$$\{m^*X \mid A \subseteq X,\ X \in \mathcal{A}\}.$$

This proves (4). □

Problems on Premeasures and Related Topics

1. Fill in the missing details in the proofs, notes, and examples of this section.

2. Describe m^* on 2^S induced by a premeasure $\mu\colon \mathcal{C} \to E^*$ such that each of the following hold.

(a) $\mathcal{C} = \{S, \emptyset\}$, $\mu S = 1$.

(b) $\mathcal{C} = \{S, \emptyset$, and all singletons$\}$; $\mu S = \infty$, $\mu\{x\} = 1$.

(c) $\mathcal{C}$ as in (b), with S uncountable; $\mu S = 1$, and $\mu X = 0$ otherwise.

(d) $\mathcal{C} = \{$all *proper* subsets of $S\}$; $\mu X = 1$ when $\emptyset \subset X \subset S$; $\mu\emptyset = 0$.

3. Show that the premeasures

$$v' : \mathcal{C}' \to [0, \infty]$$

induce *one and the same* (Lebesgue) outer measure m^* in E^n, with $v' = v$ (volume, *as in* §2):

(a) $\mathcal{C}' = \{$open intervals$\}$;

(b) $\mathcal{C}' = \{$half-open intervals$\}$;

(c) $\mathcal{C}' = \{$closed intervals$\}$;

(d) $\mathcal{C}' = \mathcal{C}_\sigma$;

(e) $\mathcal{C}' = \{$open sets$\}$;

(f) $\mathcal{C}' = \{$half-open cubes$\}$.

[Hints: (a) Let m' be the v'-induced outer measure; let $\mathcal{C} = \{$all intervals$\}$. As $\mathcal{C}' \subseteq \mathcal{C}$, $m'A \geq m^*A$. (Why?) Also,

$$(\forall\, \varepsilon > 0)\ (\exists\, \{B_k\} \subseteq \mathcal{C}) \quad A \subseteq \bigcup_k B_k \text{ and } \sum vB_k \leq m^*A + \varepsilon.$$

(Why?) By Lemma 1 in §2,

$$(\exists\, \{C_k\} \subseteq \mathcal{C}') \quad B_k \subseteq C_k \text{ and } vB_k + \frac{\varepsilon}{2^k} > v'C_k.$$

Deduce that $m^*A \geq m'A$, $m^* = m'$. Similarly for (b) and (c). For (d), use Corollary 1 and Note 3 in §1. For (e), use Lemma 2 in §2. For (f), use Problem 2 in §2.]

3′. Do Problem 3(a)–(c), with m^* replaced by the *Jordan outer content* c^* (Note 6).

4. Do Problem 3, with v and m^* replaced by the LS premeasure and outer measure. (Use Problem 7 in §4.)

5. Show that a set $A \subseteq E^n$ is bounded iff its outer Jordan content is finite.

6. Find a set $A \subseteq E^1$ such that

(i) its Lebesgue outer measure is 0 ($m^*A = 0$), while its Jordan outer content $c^*A = \infty$;

(ii) $m^*A = 0$, $c^*A = 1$ (see Corollary 6 in §2).

7. Let

$$\mu_1, \mu_2 : \mathcal{C} \to [0, \infty]$$

be two premeasures in S and let m_1^* and m_2^* be the outer measures induced by them.

Prove that if $m_1^* = m_2^*$ on $\mathcal{C}$, then $m_1^* = m_2^*$ on all of 2^S.

8. With the notation of Definition 3 and Note 6, prove the following.

(i) If $A \subseteq B \subseteq S$ and $m^*B = 0$, then $m^*A = 0$; similarly for c^*.
[Hint: Use monotonicity.]

(ii) The set family

$$\{X \subseteq S \mid c^*A = 0\}$$

is a *hereditary* set ring, i.e., a ring $\mathcal{R}$ such that

$$(\forall\, B \in \mathcal{R})\ (\forall\, A \subseteq B) \quad A \in \mathcal{R}.$$

(iii) The set family

$$\{X \subseteq S \mid m^*X = 0\}$$

is a *hereditary* σ*-ring*.

(iv) So also is

$$\mathcal{H} = \{\text{those } X \subseteq S \text{ that have basic coverings}\};$$

thus $\mathcal{H}$ is the hereditary σ-ring *generated* by $\mathcal{C}$ (see Problem 14 in §3).

9. Continuing Problem 8(iv), prove that if μ is σ-finite (Definition 4), so is m^* when restricted to $\mathcal{H}$.

Show, moreover, that if $\mathcal{C}$ is a semiring, then each $X \in \mathcal{H}$ has a basic covering $\{Y_n\}$, with $m^*Y_n < \infty$ and with all Y_n *disjoint*.
[Hint: Show that

$$X \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} B_{nk}$$

for some sets $B_{nk} \in \mathcal{C}$, with $\mu B_{nk} < \infty$. Then use Note 4 in §5 and Corollary 1 of §1.]

10. Show that if

$$s : \mathcal{C} \to E^*$$

is σ-finite and additive on $\mathcal{C}$, a *semiring*, then the σ-ring $\mathcal{R}$ generated by $\mathcal{C}$ equals the σ-ring $\mathcal{R}'$ generated by

$$\mathcal{C}' = \{X \in \mathcal{C} \mid |sX| < \infty\}$$

(cf. Problem 6 in §4).
[Hint: By σ-finiteness,

$$(\forall\, X \in \mathcal{C})\ (\exists\, \{A_n\} \subseteq \mathcal{C} \mid |sA_n| < \infty) \quad X \subseteq \bigcup_n A_n;$$

so

$$X = \bigcup_n (X \cap A_n), \quad X \cap A_n \in \mathcal{C}'.$$

(Use Lemma 3 in §4.)

Thus $(\forall\, X \in \mathcal{C})$ X is a countable union of $\mathcal{C}'$-sets; so $\mathcal{C} \subseteq \mathcal{R}'$. Deduce $\mathcal{R} \subseteq \mathcal{R}'$. Proceed.]

11. With all as in Theorem 3, prove that if A *has* basic coverings, then

$$(\exists\, B \in \mathcal{A}_\delta) \quad A \subseteq B \text{ and } m^*A = m^*B.$$

[Hint: By formula (4),

$$(\forall\, n \in N)\ (\exists\, X_n \in \mathcal{A} \mid A \subseteq X_n) \quad m^*A \le mX_n \le m^*A + \frac{1}{n}.$$

(Explain!) Set

$$B = \bigcap_{n=1}^{\infty} X_n \in \mathcal{A}_\delta.$$

Proceed. For $\mathcal{A}_\delta$, see Definition 2(b) in §3.]

12. Let $(S, \mathcal{C}, \mu)$ and m^* be as in Definition 3. Show that if $\mathcal{C}$ is a σ-field in S, then

$$(\forall\, A \subseteq S)\ (\exists\, B \in \mathcal{C}) \quad A \subseteq B \text{ and } m^*A = \mu B.$$

[Hint: Use Problem 11 and Note 3.]

⇒*13. Show that if

$$s : \mathcal{C} \to E$$

is σ-finite and σ-additive on $\mathcal{C}$, a semiring, then s has *at most one* σ-additive extension to the σ-ring $\mathcal{R}$ generated by $\mathcal{C}$.

(Note that s is automatically σ-finite if it is *finite*, e.g., complex or vector valued.)

[Outline: Let

$$s', s'' : \mathcal{R} \to E$$

be *two* σ-additive extensions of s. By Problem 10, $\mathcal{R}$ is also generated by

$$\mathcal{C}' = \{X \in \mathcal{C} \mid |sX| < \infty\}.$$

Now set

$$\mathcal{R}^* = \{X \in \mathcal{R} \mid s'X = s''X\}.$$

Show that $\mathcal{R}^*$ satisfies properties (i)–(iii) of Theorem 3 in §3, with $\mathcal{C}$ replaced by $\mathcal{C}'$; so $\mathcal{R} = \mathcal{R}^*$.]

14. Let m_n^* $(n = 1, 2, \ldots)$ be outer measures in S such that

$$(\forall\, X \subseteq S)\ (\forall\, n) \quad m_n^*X \le m_{n+1}^*X.$$

Set

$$\mu^* = \lim_{n\to\infty} m_n^*.$$

Show that μ^* is an *outer measure* in S (see Note 5).

15. An outer measure m^* in a metric space (S, ρ) is said to have the *Carathéodory property* (CP) iff

$$m^*(X \cup Y) \geq m^*X + m^*Y$$

whenever $\rho(X, Y) > 0$, where

$$\rho(X, Y) = \inf\{\rho(x, y) \mid x \in X, y \in Y\}.$$

For such m^*, prove that

$$m^*\left(\bigcup_k X_k\right) = \sum_k m^*X_k$$

if $\{X_k\} \subseteq 2^S$ and

$$\rho(X_i, X_k) > 0 \quad (i \neq k).$$

[Hint: For *finite* unions, use the CP, subadditivity, and induction. Deduce that

$$(\forall n) \quad \sum_{k=1}^{n} m^*X_k \leq m^* \bigcup_{k=1}^{\infty} X_k.$$

Let $n \to \infty$. Proceed.]

16. Let $(S, \mathcal{C}, \mu)$ and m^* be as in Definition 3, with ρ a metric for S. Let μ_n be the restriction of μ to the family $\mathcal{C}_n$ of all $X \in \mathcal{C}$ of diameter

$$dX \leq \frac{1}{n}.$$

Let m_n^* be the μ_n-induced outer measure in S.

Prove that

(i) $\{m_n^*\}\uparrow$ as in Problem 14;

(ii) the outer measure

$$\mu^* = \lim_{n \to \infty} m_n^*$$

has the CP (see Problem 15), and

$$\mu^* \geq m^* \text{ on } 2^S.$$

[Outline: Let $\rho(X, Y) > \varepsilon > 0$ $(X, Y \subseteq S)$.

If for some n, $X \cup Y$ has no basic covering from $\mathcal{C}_n$, then

$$\mu^*(X \cup Y) \geq m_n^*(X \cup Y) = \infty \geq \mu^*X + \mu^*Y,$$

and the CP follows. (Explain!)

Thus assume

$$\left(\forall n > \frac{1}{\varepsilon}\right) \; (\forall k) \; (\exists B_{nk} \in \mathcal{C}_n) \quad X \cup Y \subseteq \bigcup_{k=1}^{\infty} B_{nk}.$$

One can choose the B_{nk} so that

$$\sum_{k=1}^{\infty} \mu B_{nk} \leq m_n^*(X \cup Y) + \varepsilon.$$

(Why?) As

$$dB_{nk} \leq \frac{1}{n} < \varepsilon,$$

some B_{nk} cover X *only*, others Y *only*. (Why?) Deduce that

$$\left(\forall\, n > \frac{1}{\varepsilon}\right) \quad m_n^* X + m_n^* Y \leq \sum_{k=1}^{\infty} \mu_n B_{nk} \leq m_n^*(X \cup Y) + \varepsilon.$$

Let $\varepsilon \to 0$ and *then* $n \to \infty$.

Also, $m^* \leq m_n^* \leq \mu^*$. (Why?)]

17. Continuing Problem 16, suppose that

$$(\forall\, \varepsilon > 0)\ (\forall\, n, k)\ (\forall\, B \in \mathcal{C})\ (\exists\, B_{nk} \in \mathcal{C}_n)$$
$$B \subseteq \bigcup_{k=1}^{\infty} B_{nk} \text{ and } \mu B + \varepsilon \geq \sum_{k=1}^{\infty} \mu B_{nk}.$$

Show that

$$m^* = \lim_{n \to \infty} \mu_n^* = \mu^*,$$

so m^* *itself has the CP.*

[Hints: It suffices to prove that $m^* A \geq \mu^* A$ *if* $m^* A < \infty$. (Why?)

Now, given $\varepsilon > 0$, A has a covering

$$\{B_i\} \subseteq \mathcal{C}$$

such that

$$m^* A + \varepsilon \geq \sum \mu B_i.$$

(Why?) By assumption,

$$(\forall\, n) \quad B_i \subseteq \bigcup_{k=1}^{\infty} B_{nk}^i \in \mathcal{C}_n \text{ and } \mu B_i + \frac{\varepsilon}{2^i} \geq \sum_{k=1}^{\infty} \mu B_{nk}^i.$$

Deduce that

$$m^* A + \varepsilon > \sum \mu B_i \geq \sum_{i=1}^{\infty} \left(\sum_{k=1}^{\infty} \mu B_{nk}^i - \frac{\varepsilon}{2^i} \right) = \sum_{i,k} \mu B_{nk}^i - \varepsilon \geq m_n^* A - \varepsilon.$$

Let $\varepsilon \to 0$; *then* $n \to \infty$.]

18. Using Problem 17, show that the Lebesgue and Lebesgue–Stieltjes outer measures have the CP.

§6. Measure Spaces. More on Outer Measures[1]

I. In §5, we considered premeasure spaces, stressing mainly the idea of *σ-subadditivity* (Note 5 in §5). Now we shall emphasize *σ-additivity*.

Definition 1.

A premeasure

$$m\colon \mathcal{M} \to [0, \infty]$$

is called a *measure* (in S) iff $\mathcal{M}$ is a *σ-ring* (in S), and m is *σ-additive* on $\mathcal{M}$.

If so, the system

$$(S, \mathcal{M}, m)$$

is called a *measure space*; mX is called the *measure of* $X \in \mathcal{M}$; $\mathcal{M}$-sets are called *m-measurable* sets.

Note that m *is nonnegative* and $m\emptyset = 0$, as m is a premeasure (Definition 2 in §5).

Corollary 1. *Measures are σ-additive, σ-subadditive, monotone, and continuous.*

Proof. Use Corollary 2 in §5 and Theorem 2 in §4, noting that $\mathcal{M}$ is a *σ-ring*. □

Corollary 2. *In any measure space $(S, \mathcal{M}, m)$, the union and intersection of any sequence of m-measurable sets is m-measurable itself. So also is $X - Y$ if $X, Y \in \mathcal{M}$.*

This is obvious since $\mathcal{M}$ is a *σ-ring*.

As measures and other premeasures are *understood* to be ≥ 0, we often write

$$m\colon \mathcal{M} \to E^*$$

for

$$m\colon \mathcal{M} \to [0, \infty].$$

We also briefly say "measurable" for "m-measurable."

Note that $\emptyset \in \mathcal{M}$, but not always $S \in \mathcal{M}$.

Examples.

(a) The volume of intervals in E^n is a σ-additive premeasure, but *not a measure* since its domain (the intervals) is not a σ-ring.

(b) Let $\mathcal{M} = 2^S$. Define

$$(\forall\, X \subseteq S) \quad mX = 0.$$

---

[1] Sections 6–12 are not needed for a "limited approach." (Pass to Chapter 8, §1.)

Then m is trivially a measure (the *zero-measure*). Here *each* set $X \subseteq S$ is measurable, with $mX = 0$.

(c) Let again $\mathcal{M} = 2^S$. Let mX be the number of elements in X, if finite, and $mX = \infty$ otherwise.

Then m is a measure ("*counting measure*"). Verify!

(d) Let $\mathcal{M} = 2^S$. Fix some $p \in S$. Let

$$mX = \begin{cases} 1 & \text{if } p \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Then m is a measure (it describes a "unit mass" concentrated at p).

(e) A *probability space* is a measure space $(S, \mathcal{M}, m)$, with

$$S \in \mathcal{M} \text{ and } mS = 1.$$

In probability theory, measurable sets are called *events*; mX is called the *probability* of X, often denoted by pX or similar symbols.

In Examples (b), (c), and (d),

$$\mathcal{M} = 2^S \ (\textit{all} \text{ subsets of } S).$$

More often, however,

$$\mathcal{M} \neq 2^S,$$

i.e., there are *nonmeasurable* sets $X \subseteq S$ for which mX is not defined.

Of special interest are sets $X \in \mathcal{M}$, *with* $mX = 0$, *and their subsets*. We call them *m-null* or *null sets*. One would like them to be measurable, but this is not always the case for *subsets* of X.

This leads us to the following definition.

Definition 2.

A measure $m : \mathcal{M} \to E^*$ is called *complete* iff all null sets (subsets of sets of measure zero) are measurable.

We now develop a general method for constructing complete measures.

II. From §5 (Note 5) recall that an *outer measure* in S is a σ-subadditive premeasure defined on *all of* 2^S (even if it is not derived via Definition 3 in §5).[2] In Examples (b), (c), and (d), m is both a measure and an outer measure. (Why?)

An outer measure

$$m^* : 2^S \to E^*$$

---

[2] Some authors consider outer measures on smaller domains; we shall not do so.

need not be additive; but consider this fact:

Any set $A \subseteq S$ splits S into two parts: A itself and $-A$.

It also splits any other set X into $X \cap A$ and $X - A$; indeed,

$$X = (X \cap A) \cup (X - A) \ (\textit{disjoint}).$$

We want to single out those sets A for which m^* behaves "additively," i.e., so that

$$m^* X = m^*(X \cap A) + m^*(X - A).$$

This motivates our next definition.

Definition 3.

Given an outer measure $m^* \colon 2^S \to E^*$ and a set $A \subseteq S$, we say that A is *m^*-measurable* iff all sets $X \subseteq S$ are split "additively" by A; that is,

$$(\forall X \subseteq S) \quad m^* X = m^*(X \cap A) + m^*(X - A). \tag{1}$$

As is easily seen (see Problem 1), this is equivalent to

$$(\forall X \subseteq A)\ (\forall Y \subseteq -A) \quad m^*(X \cup Y) = m^* X + m^* Y. \tag{2}$$

The family of all m^*-measurable sets is usually denoted by $\mathcal{M}^*$. The system $(S, \mathcal{M}^*, m^*)$ is called an *outer measure space*.

Note 1. Definition 3 applies to *outer* measures only. For *measures*, "m-measurable" means simply "*member of the domain of* m" (Definition 1).

Note 2. In (1) and (2), we may equivalently replace the equality sign $(=)$ by $(\geq)$. Indeed, X is covered by

$$\{X \cap A, X - A\},$$

and $X \cup Y$ is covered by $\{X, Y\}$; so the reverse inequality $(\leq)$ anyway holds, *by subadditivity*.

Our main objective is to prove the following fundamental theorem.

Theorem 1. *In any outer measure space*

$$(S, \mathcal{M}^*, m^*),$$

the family $\mathcal{M}^*$ of all m^*-measurable sets is a σ-field in S, and m^*, when restricted to $\mathcal{M}^*$, is a complete measure (*denoted by m and called the m^*-induced measure*; *so $m^* = m$ on $\mathcal{M}^*$*).

We split the proof into several steps (lemmas).

Lemma 1. *$\mathcal{M}^*$ is closed under complementation*:

$$(\forall A \in \mathcal{M}^*) \quad -A \in \mathcal{M}^*.$$

Indeed, the measurability criterion (2) is *same* for A and $-A$ alike.

Lemma 2. *$\emptyset$ and S are $\mathcal{M}^*$-sets. So are all sets of outer measure* 0.

Proof. Let $m^*A = 0$. To prove $A \in \mathcal{M}^*$, use (2) and Note 2.

Thus take any $X \subseteq A$ and $Y \subseteq -A$. Then by monotonicity,

$$m^*X \le m^*A = 0$$

and

$$m^*Y \le m^*(X \cup Y).$$

Thus

$$m^*X + m^*Y = 0 + m^*Y \le m^*(X \cup Y),$$

as required.

In particular, as $m^*\emptyset = 0$, $\emptyset$ is m^*-measurable ($\emptyset \in \mathcal{M}^*$).

So is S (the complement of $\emptyset$) by Lemma 1. □

Lemma 3. *$\mathcal{M}^*$ is closed under finite unions*:

$$(\forall A, B \in \mathcal{M}^*) \quad A \cup B \in \mathcal{M}^*.$$

Proof. This time we shall use formula (1). By Note 2, it suffices to show that

$$(\forall X \subseteq S) \quad m^*X \ge m^*(X \cap (A \cup B)) + m^*(X - (A \cup B)).$$

Fix any $X \subseteq S$; as $A \in \mathcal{M}^*$, we have

$$m^*X = m^*(X \cap A) + m^*(X - A). \tag{3}$$

Similarly, as $B \in \mathcal{M}^*$, we have (replacing X by $X - A$ in (1))

$$\begin{aligned} m^*(X - A) &= m^*((X - A) \cap B) + m^*(X - A - B) \\ &= m^*(X \cap -A \cap B) + m^*(X - (A \cup B)), \end{aligned} \tag{4}$$

since

$$X - A = X \cap -A$$

and

$$X - A - B = X - (A \cup B).$$

Combining (4) with (3), we get

$$m^*X = m^*(X \cap A) + m^*(X \cap -A \cap B) + m^*(X - (A \cup B)). \tag{5}$$

Now verify that

$$(X \cap A) \cup (X \cap -A \cap B) \supseteq X \cap (A \cup B).$$

As m is subadditive, this yields

$$m^*(X \cap A) + m^*(X \cap -A \cap B) \geq m^*(X \cap (A \cup B)).$$

Combining with (5), we get

$$m^* X \geq m^*(X \cap (A \cup B)) + m^*(X - (A \cup B)),$$

so that $A \cup B \in \mathcal{M}^*$, indeed. □

Induction extends Lemma 3 to all finite unions of $\mathcal{M}^*$-sets.

Note that by Problem 3 in §3, $\mathcal{M}^*$ is a *set field*, hence surely a ring. Thus Corollary 1 in §1 applies to it. (We use it below.)

Lemma 4. *Let*

$$X_k \subseteq A_k \subseteq S, \quad k = 0, 1, 2, \ldots,$$

with all A_k pairwise disjoint.

Let $A_k \in \mathcal{M}^*$ for $k \geq 1$. (A_0 and the X_k need not be $\mathcal{M}^*$-sets.) Then

$$m^*\left(\bigcup_{k=0}^{\infty} X_k\right) = \sum_{k=0}^{\infty} m^* X_k. \tag{6}$$

Proof. We start with *two* sets, A_0 and A_1; so

$$A_1 \in \mathcal{M}^*,\ A_0 \cap A_1 = \emptyset,\ X_0 \subseteq A_0, \text{ and } X_1 \subseteq A_1.$$

As $A_0 \cap A_1 = \emptyset$, we have $A_0 \subseteq -A_1$; hence also $X_0 \subseteq -A_1$.

Since $A_1 \in \mathcal{M}^*$, we use formula (2), with

$$X = X_1 \subseteq A_1 \text{ and } Y = X_0 \subseteq -A,$$

to obtain

$$m^*(X_0 \cup X_1) = m^* X_0 + m^* X_1.$$

Thus (6) holds for *two* sets.

Induction now easily yields

$$(\forall n) \quad \sum_{k=0}^{n} m^* X_k = m^*\left(\bigcup_{k=0}^{n} X_k\right) \leq m^*\left(\bigcup_{k=0}^{\infty} X_k\right)$$

by monotonicity of m^*. Now let $n \to \infty$ and pass to the limit to get

$$\sum_{k=0}^{\infty} m^* X_k \leq m^*\left(\bigcup_{k=0}^{\infty} X_k\right).$$

As $\bigcup X_k$ is *covered* by the X_k, the σ-subadditivity of m^* yields the reverse inequality as well. Thus (6) is proved. □

Proof of Theorem 1. As we noted, $\mathcal{M}^*$ is a field. To show that it is also closed under *countable* unions (a *σ-field*), let

$$U = \bigcup_{k=1}^{\infty} A_k, \quad A_k \in \mathcal{M}^*.$$

We have to prove that $U \in \mathcal{M}^*$; or by (2) and Note 2,

$$(\forall X \subseteq U)\ (\forall Y \subseteq -U) \quad m^*(X \cup Y) \geq m^*X + m^*Y. \tag{7}$$

We may safely assume that the A_k are *disjoint*. (If not, replace them by disjoint sets $B_k \in \mathcal{M}^*$, as in Corollary 1 of §1.)

To prove (7), fix any $X \subseteq U$ and $Y \subseteq -U$, and let

$$X_k = X \cap A_k \subseteq A_k,$$

$A_0 = -U$, and $X_0 = Y$, satisfying all assumptions of Lemma 4. Thus by (6), writing the first term separately, we have

$$m^*\left(Y \cup \bigcup_{k=1}^{\infty} X_k\right) = m^*Y + \sum_{k=1}^{\infty} m^*X_k. \tag{8}$$

But

$$\bigcup_{k=1}^{\infty} X_k = \bigcup_{k=1}^{\infty} (X \cap A_k) = X \cap \bigcup_{k=1}^{\infty} A_k = X \cap U = X$$

(as $X \subseteq U$). Also, by σ-subadditivity,

$$\sum m^*X_k \geq m^* \bigcup X_k = m^*X.$$

Therefore, (8) implies (7); so $\mathcal{M}^*$ is a *σ-field*.

Moreover, m^* is *σ-additive* on $\mathcal{M}^*$, as follows from Lemma 4 by taking

$$X_k = A_k \in \mathcal{M}^*,\ A_0 = \emptyset.$$

Thus m^* acts as a *measure* on $\mathcal{M}^*$.

By Lemma 2, m^* is *complete*; for if X is "null" ($X \subseteq A$ and $m^*A = 0$), then $m^*X = 0$; so $X \in \mathcal{M}^*$, as required.

Thus all is proved. □

We thus have a standard method for constructing measures: From a premeasure

$$\mu\colon \mathcal{C} \to E^*$$

in S, we obtain the μ-induced *outer measure*

$$m^*\colon 2^S \to E^* \text{ (§5)};$$

this, in turn, induces a complete *measure*

$$m\colon \mathcal{M}^* \to E^*.$$

But we need more: We want m to be an *extension* of μ, i.e.,

$$m = \mu \text{ on } \mathcal{C},$$

with $\mathcal{C} \subseteq \mathcal{M}^*$ (meaning that *all $\mathcal{C}$-sets are m^*-measurable*). We now explore this question.

Lemma 5. *Let $(S, \mathcal{C}, \mu)$ and m^* be as in Definition 3 of §5. Then for a set $A \subseteq S$ to be m^*-measurable, it suffices that*

$$m^*X \geq m^*(X \cap A) + m^*(x - A) \quad \textit{for all } X \in \mathcal{C}. \tag{9}$$

Proof. Assume (9). We must show that (9) holds for *any* $X \subseteq S$, even not a $\mathcal{C}$-set.

This is trivial if $m^*X = \infty$. Thus assume $m^*X < \infty$ and fix any $\varepsilon > 0$.

By Note 3 in §5, X *must* have a basic covering $\{B_n\} \subseteq \mathcal{C}$ so that

$$X \subseteq \bigcup_n B_n$$

and

$$m^*X + \varepsilon > \sum \mu B_n \geq \sum m^*B_n. \tag{10}$$

(Explain!)

Now, as $X \subseteq \bigcup B_n$, we have

$$X \cap A \subseteq \bigcup B_n \cap A = \bigcup (B_n \cap A).$$

Similarly,

$$X - A = X \cap -A \subseteq \bigcup (B_n - A).$$

Hence, as m^* is σ-subadditive and monotone, we get

$$\begin{aligned} m^*(X \cap A) + m^*(X - A) &\leq m^*\Big(\bigcup (B_n \cap A)\Big) + m^*\Big(\bigcup (B_n - A)\Big) \\ &\leq \sum [m^*(B_n \cap A) + m^*(B_n - A)]. \end{aligned} \tag{11}$$

But by assumption, (9) holds for any $\mathcal{C}$-*set*, hence for each B_n. Thus

$$m^*(B_n \cap A) + m^*(B_n - A) \leq m^*B_n,$$

and (11) yields

$$m^*(X \cap A) + m^*(X - A) \leq \sum [m^*(B_n \cap A) + m^*(B_n - A)] \leq \sum m^*B_n.$$

Therefore, by (10),

$$m^*(X \cap A) + m^*(X - A) \le m^*X + \varepsilon.$$

Making $\varepsilon \to 0$, we prove (10) for *any* $X \subseteq S$, so that $A \in \mathcal{M}^*$, as required. $\square$

Theorem 2. *Let the premeasure*

$$\mu \colon \mathcal{C} \to E^*$$

be σ-additive on $\mathcal{C}$, a semiring *in S. Let m^* be the μ-induced outer measure, and*

$$m \colon \mathcal{M}^* \to E^*$$

be the m^*-induced measure. Then

(i) $\mathcal{C} \subseteq \mathcal{M}^*$ *and*

(ii) $\mu = m^* = m$ *on* $\mathcal{C}$.

Thus m is a σ-additive extension of μ (called its Lebesgue extension*) to $\mathcal{M}^*$.*

Proof. By Corollary 2 in §5, μ is also σ-*sub*additive on the semiring $\mathcal{C}$. Thus by Theorem 2 in §5, $\mu = m^*$ on $\mathcal{C}$.

To prove that $\mathcal{C} \subseteq \mathcal{M}^*$, we fix $A \in \mathcal{C}$ and show that A satisfies (9), so that $A \in \mathcal{M}^*$.

Thus take any $X \in \mathcal{C}$. As $\mathcal{C}$ is a semiring, $X \cap A \in \mathcal{C}$ and

$$X - A = \bigcup_{k=1}^{n} A_k \ (\textit{disjoint})$$

for some sets $A_k \in \mathcal{C}$. Hence

$$\begin{aligned} m^*(X \cap A) + m^*(X - A) &= m^*(X \cap A) + m^* \bigcup_{k=1}^{n} A_k \\ &\le m^*(X \cap A) + \sum_{k=1}^{n} m^* A_k. \end{aligned} \tag{12}$$

As

$$X = (X \cap A) \cup (X - A) = (X \cap A) \cup \bigcup A_k \ (\textit{disjoint}),$$

the additivity of μ and the equality $\mu = m^*$ on $\mathcal{C}$ yield

$$m^*X = m^*(X \cap A) + \sum_{k=1}^{n} m^* A_k.$$

Hence by (12),

$$m^*X \ge m^*(X \cap A) + m^*(X - A);$$

so by Lemma 5, $A \in \mathcal{M}^*$, as required.

Also, by definition, $m = m^*$ on $\mathcal{M}^*$, hence on $\mathcal{C}$. Thus

$$\mu = m^* = m \text{ on } \mathcal{C},$$

as claimed. $\square$

Note 3. In particular, Theorem 2 applies if

$$\mu \colon \mathcal{M} \to E^*$$

is a *measure* (so that $\mathcal{C} = \mathcal{M}$ is even a σ*-ring*).

Thus *any such* μ *can be extended to a complete measure* m (its Lebesgue extension) on a σ-field

$$\mathcal{M}^* \supseteq \mathcal{M}$$

via the μ-induced outer measure (call it μ^* this time), with

$$\mu^* = m = \mu \text{ on } \mathcal{M}.$$

Moreover,

$$\mathcal{M}^* \supseteq \mathcal{M} \supseteq \mathcal{M}_\sigma$$

(see Note 2 in §3); so μ^* *is* $\mathcal{M}$*-regular and* $\mathcal{M}^*$*-regular* (Theorem 3 of §5).

Note 4. A reapplication of this process *to* m does not change m (Problem 16).

Problems on Measures and Outer Measures

1. Show that formulas (1) and (2) are equivalent.

[Hints: (i) Assume (1) and let $X \subseteq A$, $Y \subseteq -A$.

As X in (1) is arbitrary, we may replace it by $X \cup Y$. Simplifying, obtain (2) on noting that $X \cap A = X$, $X \cap -A = \emptyset$, $Y \cap A = \emptyset$, and $Y \cap -A = Y$.

(ii) Assume (2). Take any X and substitute $X \cap A$ and $X - A$ for X and Y in (2).]

2. Given an outer measure space $(S, \mathcal{M}^*, m^*)$ and $A \subseteq S$, set

$$A \capdot \mathcal{M}^* = \{A \cap X \mid X \in \mathcal{M}^*\}$$

(all sets of the form $A \cap X$ with $X \in \mathcal{M}^*$).

Prove that $A \capdot \mathcal{M}^*$ is a σ-field *in* A, and m^* is σ-additive on it.

[Hint: Use Lemma 4, with $X_k = A \cap A_k \in A \capdot \mathcal{M}^*$.]

3. Prove Lemmas 1 and 2, using *formula* (1).

3′. Prove Corollary 1.

4. Verify Examples (b), (c), and (d). Why is m an *outer* measure as well?

[Hint: Use Corollary 2 in §5.]

5. Fill in all details (induction, etc.) in the proofs of this section.

6. Verify that m^* is an outer measure and describe $\mathcal{M}^*$ under each of the following conditions.

(a) $m^*A = 1$ if $\emptyset \subset A \subseteq S$; $m^*\emptyset = 0$.

(b) $m^*A = 1$ if $\emptyset \subset A \subset S$; $m^*S = 2$; $m^*\emptyset = 0$.

(c) $m^*A = 0$ if $A \subseteq S$ is countable; $m^*A = 1$ otherwise (S is uncountable).

(d) $S = N$ (naturals); $m^*A = 1$ if A is infinite; $m^*A = \frac{n}{n+1}$ if A has n elements.

7. Prove the following.

(i) An outer measure m^* is $\mathcal{M}^*$-regular (Definition 5 in §5) iff

$$(\forall\, A \subseteq S)\ (\exists\, B \in \mathcal{M}^*) \quad A \subseteq B \text{ and } m^*A = mB.$$

B is called a *measurable cover* of A.
[Hint: If

$$m^*A = \inf\{mX \mid A \subseteq X \in \mathcal{M}^*\},$$

then

$$(\forall\, n)\ (\exists\, X_n \in \mathcal{M}^*) \quad A \subseteq X_n \text{ and } mX_n \le m^*A + \frac{1}{n}.$$

Set $B = \bigcap_{n=1}^{\infty} X_n$.]

(ii) If m^* is as in Definition 3 of §5, with $\mathcal{C} \subseteq \mathcal{M}^*$, then m^* is $\mathcal{M}^*$-regular.

8. Show that if m^* is $\mathcal{M}^*$-regular (Problem 7), it is left continuous.
[Hints: Let $\{A_n\}\uparrow$; let B_n be a measurable cover of A_n; set

$$C_n = \bigcap_{k=n}^{\infty} B_k.$$

Verify that $\{C_n\}\uparrow$, $B_n \supseteq C_n \supseteq A_n$, and $mC_n = m^*A_n$.

By the left continuity *of* m (Theorem 2 in §4),

$$\lim m^*A_n = \lim mC_n = m \bigcup_{n=1}^{\infty} C_n \ge m^* \bigcup_{n=1}^{\infty} A_n.$$

Prove the reverse inequality as well.]

9. Continuing Problems 6–8, verify the following.

(i) In 6(a), with $S = N$, m^* is $\mathcal{M}^*$-regular, but not *right* continuous.
Hint: Take $A_n = \{x \in N \mid x \ge n\}$.

(ii) In 6(b), with $S = N$, m^* is neither $\mathcal{M}^*$-regular nor left continuous.

(iii) In 6(d), m^* is not $\mathcal{M}^*$-regular; yet it is left continuous. (Thus Problem 8 is not a *necessary* condition.)

10. In Problem 2, let n^* be the restriction of m^* to 2^A. Prove the following.

(a) n^* is an outer measure *in* A.

(b) $A \cap \mathcal{M}^* \subseteq \mathcal{N}^* = \{n^*\text{-measurable sets}\}$.

(c) $A \cap \mathcal{M}^* = \mathcal{N}^*$ if $A \in \mathcal{M}^*$, or if m^* is $\mathcal{M}^*$-regular (see Problem 7) and finite.

(d) n^* is $\mathcal{N}^*$-regular if m^* is $\mathcal{M}^*$-regular.

11. Show that if m^* is $\mathcal{M}^*$-regular and finite, then $A \subseteq S$ is m^*-measurable *iff*

$$mS = m^*A + m^*(-A).$$

[Hint: Assume the latter. By Problem 7,

$$(\forall X \subseteq S)\ (\exists B \in \mathcal{M}^*,\ B \supseteq X) \quad m^*X = mB;$$

so

$$m^*A = m^*(A \cap B) + m^*(A - B).$$

Similarly for $-A$. Deduce that

$$m^*(A \cap B) + m^*(A - B) + m^*(B - A) + m^*(-A - B) = mS = mB + m(-B);$$

hence

$$m^*X = mB \geq m^*(B \cap A) + m^*(B - A) \geq m^*(X \cap A) + m^*(X - A),$$

so $A \in \mathcal{M}^*$.]

12. Using Problem 15 in §5, prove that *if m^* has the CP then each open set $G \subseteq S$ is in $\mathcal{M}^*$.*

[Outline: Show that

$$(\forall X \subseteq G)\ (\forall Y \subseteq -G) \quad m^*(X \cup Y) \geq m^*X + m^*Y,$$

assuming $m^*X < \infty$. (Why?) Set

$$D_0 = \{x \in X \mid \rho(x, -G) \geq 1\}$$

and

$$D_k = \left\{x \in X \;\middle|\; \frac{1}{k+1} \leq \rho(x, -G) < \frac{1}{k}\right\}, \quad k \geq 1.$$

Prove that

(i)
$$X = \bigcup_{k=0}^{\infty} D_k$$

and

(ii)
$$\rho(D_k, D_{k+2}) > 0;$$

so by Problem 15 in §5,

$$\sum_{n=0}^{\infty} m^*D_{2n} = m^* \bigcup_{n=0}^{\infty} D_{2n} \leq m^* \bigcup_{n=0}^{\infty} D_n = m^*X < \infty.$$

Similarly,

$$\sum_{n=0}^{\infty} m^* D_{2n+1} \le m^* X < \infty.$$

Hence

$$\sum_{n=0}^{\infty} m^* D_n < \infty;$$

so

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} m^* D_k = 0.$$

(Why?) Thus

$$(\forall \varepsilon > 0)\ (\exists\, n) \quad \sum_{k=n}^{\infty} m^* D_k < \varepsilon.$$

Also,

$$X = \bigcup_{k=0}^{\infty} D_k = \bigcup_{k=0}^{n-1} D_k \cup \bigcup_{k=n}^{\infty} D_k;$$

so

$$m^* X \le m^* \bigcup_{k=0}^{n-1} D_k + \sum_{k=n}^{\infty} m^* D_k < m^* \bigcup_{k=0}^{n-1} D_k + \varepsilon.$$

Adding m^*Y on both sides, get

$$\text{(iii)} \qquad m^* X + m^* Y \le m^* \bigcup_{k=0}^{n-1} D_k + m^* Y + \varepsilon.$$

Moreover,

$$\rho\Big(\bigcup_{k=0}^{n-1} D_k, Y\Big) > 0,$$

for $Y \subseteq -G$ and

$$\rho(D_k, -G) \ge \frac{1}{k+1}.$$

Hence by the CP,

$$m^* Y + \sum_{k=0}^{n-1} m^* D_k = m^*\Big(Y \cup \bigcup_{k=0}^{n-1} D_k\Big) < m^*(Y \cup X).$$

(Why?) Combining with (iii), obtain

$$m^* X + m^* Y \le m^*(X \cup Y) + \varepsilon.$$

Now let $\varepsilon \to 0$.]

⇒13. Show that if $m\colon \mathcal{M} \to E^*$ is a measure, there is $P \in \mathcal{M}$, with

$$mP = \max\{mX \mid X \in \mathcal{M}\}.$$

[Hint: Let

$$k = \sup\{mX \mid X \in \mathcal{M}\}$$

in E^*. As $k \geq 0$, there is a sequence $r_n \nearrow k$, $r_n < k$. (If $k = \infty$, set $r_n = n$; if $k < \infty$, $r_n = k - \frac{1}{n}$.) By lub properties,

$$(\forall\, n)\ (\exists\, X_n \in \mathcal{M}) \quad r_n < mX_n \leq k,$$

with $\{X_n\}\!\uparrow$ (Problem 9 in §3). Set

$$P = \bigcup_{n=1}^{\infty} X_n.$$

Show that

$$mP = \lim_{n\to\infty} mX_n = k.]$$

⇒***14.** Given a measure $m\colon \mathcal{M} \to E^*$, let

$$\overline{\mathcal{M}} = \{\text{all sets of the form } X \cup Z \text{ where } X \in \mathcal{M} \text{ and } Z \text{ is } m\text{-}null\}.$$

Prove that $\overline{\mathcal{M}}$ is a σ-ring $\supseteq \mathcal{M}$.
[Hint: To prove that

$$(\forall\, A, B \in \overline{\mathcal{M}}) \quad A - B \in \overline{\mathcal{M}},$$

suppose first $A \in \mathcal{M}$ and B is "null," i.e., $B \subseteq U \in \mathcal{M}$, $mU = 0$.

Show that

$$A - B = X \cup Z,$$

with $X = A - U \in \mathcal{M}$ and $Z = A \cap U - B$ *m-null* (Z is *shaded* in Figure 31).

Next, if $A, B \in \overline{\mathcal{M}}$, let $A = X \cup Z$, $B = X' \cup Z'$, where $X, X' \in \mathcal{M}$ and Z, Z' are m-null. Hence

$$\begin{aligned} A - B &= (X \cup Z) - B \\ &= (X - B) \cup (Z - B) \\ &= (X - B) \cup Z'', \end{aligned}$$

where

$$Z'' = Z - B$$

is m-null. Also, $B = X' \cup Z'$ implies

$$X - B = (X - X') - Z' \in \overline{\mathcal{M}},$$

by the first part of the proof.

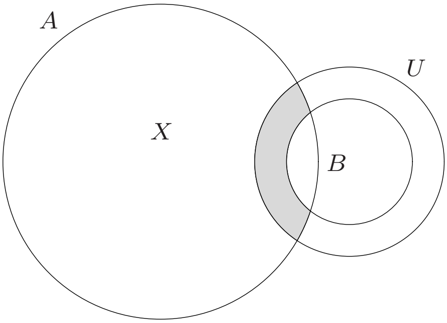

FIGURE 31

Deduce that

$$A - B = (X - B) \cup Z'' \in \overline{\mathcal{M}}$$

(after checking closure under unions).]

⇒***15.** Continuing Problem 14, define $\overline{m}\colon \overline{\mathcal{M}} \to E^*$ by setting $\overline{m}A = mX$ whenever $A = X \cup Z$, with $X \in \mathcal{M}$ and Z m-null. (Show that $\overline{m}A$ does not depend on the particular representation of A as $X \cup Z$.)

Prove the following.

(i) $\overline{m}$ is a complete measure (called the *completion* of m), with $\overline{m} = m$ on $\mathcal{M}$.

(ii) $\overline{m}$ is the *least* complete extension of m; that is, if $n\colon \mathcal{N} \to E^*$ is another complete measure, with $\mathcal{M} \subseteq \mathcal{N}$ and $n = m$ on $\mathcal{M}$, then $\overline{\mathcal{M}} \subseteq \mathcal{N}$ and $n = \overline{m}$ on $\overline{\mathcal{M}}$.

(iii) $m = \overline{m}$ iff m is complete.

***16.** Show that if $m\colon \mathcal{M}^* \to E^*$ is induced by an $\mathcal{M}^*$-*regular* outer measure μ^*, then m *equals* its Lebesgue extension m' and completion $\overline{m}$ (see Problem 15).

[Hint: By Definition 3 in §5, m induces an outer measure m^*. By Theorem 3 in §5,

$$m^*A = \inf\{mX \mid A \subseteq X \in \mathcal{M}^*\} = \mu^*A$$

(for μ^* is $\mathcal{M}^*$-*regular*).

As $m^* = \mu^*$, we get $m' = m$. Also, $m = \overline{m}$, by Problem 15(iii).]

***17.** Prove that if a measure $\mu\colon \mathcal{M} \to E^*$ is σ-finite (Definition 4 in §5), with $S \in \mathcal{M}$, then its Lebesgue extension $m\colon \mathcal{M}^* \to E^*$ equals its completion $\overline{\mu}$ (see Problem 15).

[Outline: It suffices to prove $\mathcal{M}^* \subseteq \overline{\mathcal{M}}$. (Why?)

To start with, let $A \in \mathcal{M}^*$, $mA < \infty$. By Problem 12 in §5,

$$(\exists\, B \in \mathcal{M}) \quad A \subseteq B \text{ and } m^*A = mA = mB < \infty;$$

so

$$m(B - A) = mB - mA = 0.$$

Also,

$$(\exists\, H \in \mathcal{M}) \quad B - A \subseteq H \text{ and } \mu H = m(B - A) = 0.$$

Thus $B - A$ is μ-*null*; so $B - A \in \overline{\mathcal{M}}$. (Why?) Deduce that

$$A = B - (B - A) \in \overline{\mathcal{M}}.$$

Thus $\overline{\mathcal{M}}$ contains any $A \in \mathcal{M}^*$ with $mA < \infty$. Use the σ-finiteness of μ to show

$$(\forall\, x \in \mathcal{M}^*)\ (\exists\, \{A_n\} \subseteq \mathcal{M}^*) \quad mA_n < \infty \text{ and } X = \bigcup_n A_n \in \overline{\mathcal{M}}.]$$

§7. Topologies. Borel Sets. Borel Measures

I. Our theory of set families leads quite naturally to a generalization of metric spaces. As we know, in any such space (S, ρ), there is a family $\mathcal{G}$ of *open* sets, and a family $\mathcal{F}$ of all *closed* sets. In Chapter 3, §12, we derived the following two properties.

(i) $\mathcal{G}$ is closed under *any* (even uncountable) unions and under finite intersections (Chapter 3, §12, Theorem 2). Moreover,

$$\emptyset \in \mathcal{G} \text{ and } S \in \mathcal{G}.$$

(ii) $\mathcal{F}$ has these properties, with "unions" and "intersections" *interchanged* (Chapter 3, §12, Theorem 3). Moreover, by definition,

$$A \in \mathcal{F} \text{ iff } -A \in \mathcal{G}.$$

Now, quite often, it is not so important to have *distances* (i.e., a *metric*) defined in S, but rather to single out two set families, $\mathcal{G}$ and $\mathcal{F}$, with properties (i) and (ii), in a suitable manner. For examples, see Problems 1 to 4 below. Once $\mathcal{G}$ and $\mathcal{F}$ are given, one does not need a metric to define such notions as continuity, limits, etc. (See Problems 2 and 3.) This leads us to the following definition.

Definition 1.

A *topology* for a set S is any set family $\mathcal{G} \subseteq 2^S$, with properties (i).

The pair $(S, \mathcal{G})$ then is called a *topological space*. If confusion is unlikely, we simply write S for $(S, \mathcal{G})$.

$\mathcal{G}$–sets are called *open* sets; their complements form the family $\mathcal{F}$ (called *cotopology*) of all *closed* sets in S; $\mathcal{F}$ satisfies (ii) (the proof is as in Theorem 3 of Chapter 3, §12).

Any metric space may be treated as a topological one (with $\mathcal{G}$ defined as in Chapter 3, §12), but the converse is not true. Thus $(S, \mathcal{G})$ *is more general*.

Note 1. By Problem 15 in Chapter 4, §2, *a map*

$$f\colon (S, \rho) \to (T, \rho')$$

is continuous iff $f^{-1}[B]$ *is open in* S *whenever* B *is open in* T.

We adopt this as a *definition*, for *topological* spaces S, T.

Many other notions (neighborhoods, limits, etc.) carry over from metric spaces by simply treating G_p as "*an open set containing* p." (See Problem 3.)

Note 2. By (i), $\mathcal{G}$ is surely closed under *countable* unions. Thus by Note 2 in §3,

$$\mathcal{G} = \mathcal{G}_\sigma.$$

Also, $\mathcal{G} = \mathcal{G}_d$ and

$$\mathcal{F}_\delta = \mathcal{F} = \mathcal{F}_s,$$

but not

$$\mathcal{G} = \mathcal{G}_\delta \text{ or } \mathcal{F} = \mathcal{F}_\sigma$$

in general.

$\mathcal{G}$ and $\mathcal{F}$ *need not* be rings or σ-rings (closure fails for *differences*). But by Theorem 2 in §3, $\mathcal{G}$ and $\mathcal{F}$ can be "embedded" in a smallest σ-ring. We *name* it in the following definition.

Definition 2.

The σ-ring $\mathcal{B}$ generated by a topology $\mathcal{G}$ in S is called the *Borel field* in S. (It is a *σ-field*, as $S \in \mathcal{G} \subseteq \mathcal{B}$.)

Equivalently, $\mathcal{B}$ is the least σ-ring $\supseteq \mathcal{F}$. (Why?)

$\mathcal{B}$-sets are called *Borel sets* in $(S, \mathcal{G})$.

As $\mathcal{B}$ is closed under countable unions and intersections, we have not only

$$\mathcal{B} \supseteq \mathcal{G} \text{ and } \mathcal{B} \supseteq \mathcal{F},$$

but also

$$\mathcal{B} \supseteq \mathcal{G}_\delta,\ \mathcal{B} \supseteq \mathcal{F}_\sigma,\ \mathcal{B} \supseteq \mathcal{G}_{\delta\sigma} \text{ [i.e., } (\mathcal{G}_\delta)_\sigma],\ \mathcal{B} \supseteq \mathcal{F}_{\sigma\delta}, \text{ etc.}$$

Note that

$$\mathcal{G}_{\delta\delta} = \mathcal{G}_\delta,\ \mathcal{F}_{\sigma\sigma} = \mathcal{F}_\sigma, \text{ etc. (Why?)}$$

II. Special notions apply to *measures* in metric and topological spaces.

Definition 3.

A measure $m\colon \mathcal{M} \to E^*$ in $(S, \mathcal{G})$ is called *topological* iff $\mathcal{G} \subseteq \mathcal{M}$, i.e., *all open sets are measurable*; m is a *Borel measure* iff $\mathcal{M} = \mathcal{B}$.

Note 3. *If $\mathcal{G} \subseteq \mathcal{M}$ (a σ-ring), then also $\mathcal{B} \subseteq \mathcal{M}$* since $\mathcal{B}$ is, by definition, the *least* σ-ring $\supseteq \mathcal{G}$.

Thus *m is topological iff $\mathcal{B} \subseteq \mathcal{M}$* (hence surely $\mathcal{F} \subseteq \mathcal{M}$, $\mathcal{G}_\delta \subseteq \mathcal{M}$, $\mathcal{F}_\sigma \subseteq \mathcal{M}$, etc.).

It also follows that any topological measure can be *restricted to $\mathcal{B}$* to obtain a Borel measure, called its *Borel restriction.*

Definition 4.

A measure $m\colon \mathcal{M} \to E^*$ in $(S, \mathcal{G})$ is called *regular* iff it is regular with respect to $\mathcal{M} \cap \mathcal{G}$, the *measurable open sets*; i.e.,

$$(\forall\, A \in \mathcal{M}) \quad mA = \inf\{mX \mid A \subseteq X \in \mathcal{M} \cap \mathcal{G}\}.$$

If m is topological ($\mathcal{G} \subseteq \mathcal{M}$), this simplifies to

$$mA = \inf\{mX \mid A \subseteq X \in \mathcal{G}\}, \tag{1}$$

i.e., m is $\mathcal{G}$-regular (Definition 5 in §5).

Definition 5.

A measure m is *strongly regular* iff for any $A \in \mathcal{M}$ and $\varepsilon > 0$, there is an open set $G \in \mathcal{M}$ and a closed set $F \in \mathcal{M}$ such that

$$F \subseteq A \subseteq G, \text{ with } m(A - F) < \varepsilon \text{ and } m(G - A) < \varepsilon; \tag{2}$$

thus A can be "approximated" by open supersets and closed subsets, both measurable. As is easily seen, this *implies* regularity.

A kind of converse is given by the following theorem.

Theorem 1. *If a measure $m\colon \mathcal{M} \to E^*$ in $(S, \mathcal{G})$ is regular and σ-finite (see Definition 4 in §5), with $S \in \mathcal{M}$, then m is also strongly regular.*

Proof. Fix $\varepsilon > 0$ and let $mA < \infty$.

By regularity,

$$mA = \inf\{mX \mid A \subseteq X \in \mathcal{M} \cap \mathcal{G}\};$$

so there is a set $X \in \mathcal{M} \cap \mathcal{G}$ (measurable and open), with

$$A \subseteq X \text{ and } mX < mA + \varepsilon.$$

Then

$$m(X - A) = mX - mA < \varepsilon,$$

and X is the open set G required in (2).

If, however, $mA = \infty$, use σ-finiteness to obtain

$$A \subseteq \bigcup_{k=1}^{\infty} X_k$$

for some sets $X_k \in \mathcal{M}$, $mX_k < \infty$; so

$$A = \bigcup_k (A \cap X_k).$$

Put

$$A_k = A \cap X_k \in \mathcal{M}.$$

(Why?) Then

$$A = \bigcup_k A_k,$$

and

$$mA_k \le mX_k < \infty.$$

Now, by what was proved above, for each A_k there is an open *measurable* $G_k \supseteq A_k$, with

$$m(G_k - A_k) < \frac{\varepsilon}{2^k}.$$

Set

$$G = \bigcup_{k=1}^{\infty} G_k.$$

Then $G \in \mathcal{M} \cap \mathcal{G}$ and $G \supseteq A$. Moreover,

$$G - A = \bigcup_k G_k - \bigcup_k A_k \subseteq \bigcup_k (G_k - A_k).$$

(Verify!) Thus by σ-subadditivity,

$$m(G - A) \leq \sum_k m(G_k - A_k) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon,$$

as required.

To find also the closed set F, consider

$$-A = S - A \in \mathcal{M}.$$

As shown above, there is an open measurable set $G' \supseteq -A$, with

$$\varepsilon > m(G' - (-A)) = m(G' \cap A) = m(A - (-G')).$$

Then

$$F = -G' \subseteq A$$

is the desired closed set, with $m(A - F) < \varepsilon$. $\square$

Theorem 2. *If $m\colon \mathcal{M} \to E^*$ is a strongly regular measure in $(S, \mathcal{G})$, then for any $A \in \mathcal{M}$, there are measurable sets $H \in \mathcal{F}_\sigma$ and $K \in \mathcal{G}_\delta$ such that*

$$H \subseteq A \subseteq K \text{ and } m(A - H) = 0 = m(K - A); \tag{3}$$

hence

$$mA = mH = mK.$$

Proof. Let $A \in \mathcal{M}$. By strong regularity, given $\varepsilon_n = 1/n$, one finds measurable sets

$$G_n \in \mathcal{G} \text{ and } F_n \in \mathcal{F}, \quad n = 1, 2, \ldots,$$

such that

$$F_n \subseteq A \subseteq G_n$$

and

$$m(A - F_n) < \frac{1}{n} \text{ and } m(G_n - A) < \frac{1}{n}, \quad n = 1, 2, \ldots. \tag{4}$$

Let

$$H = \bigcup_{n=1}^{\infty} F_n \text{ and } K = \bigcap_{n=1}^{\infty} G_n.$$

Then $H, K \in \mathcal{M}$, $H \in \mathcal{F}_\sigma$, $K \in \mathcal{G}_\delta$, and

$$H \subseteq A \subseteq K.$$

Also, $F_n \subseteq H$ and $G_n \supseteq K$.

Hence

$$A - H \subseteq A - F_n \text{ and } K - A \subseteq G_n - A;$$

so by (4),

$$m(A - H) < \frac{1}{n} \to 0 \text{ and } m(K - A) < \frac{1}{n} \to 0.$$

Finally,

$$mA = m(A - H) + mH = mH,$$

and similarly $mA = mK$.

Thus all is proved. □

Problems on Topologies, Borel Sets, and Regular Measures

1. Show that $\mathcal{G}$ is a topology in S (in (a)–(c), describe $\mathcal{B}$ also), given

(a) $\mathcal{G} = 2^S$;

(b) $\mathcal{G} = \{\emptyset, S\}$;

(c) $\mathcal{G} = \{\emptyset$ and all sets in S, containing a fixed point $p\}$; or

(d) $S = E^*$; $\mathcal{G}$ consists of all possible unions of sets of the form (a, b), $(a, \infty]$, and $[-\infty, b)$, with $a, b \in E^1$.

2. (S, ρ) is called a *pseudometric space* (and ρ is a *pseudometric*) iff the metric laws (i)–(iii) of Chapter 3, §11 hold, but (i′) is weakened to

$$\rho(x, x) = 0$$

(so that $\rho(x, y)$ may be 0 even if $x \neq y$).

(a) Define "globes," "interiors," and "open sets" (i.e., $\mathcal{G}$) as in Chapter 3, §12; then show that $\mathcal{G}$ is a topology for S.

(b) Let $S = E^2$ and

$$\rho(\bar{x}, \bar{y}) = |x_1 - y_1|,$$

where $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$. Show that ρ is a pseudometric but *not a metric* (the Hausdorff properly fails!).

3. Define "neighborhood," "interior," "cluster point," "closure," and "function limit" for topological spaces. Specify some notions (e.g., "diameter," "uniform continuity") that *do not* carry over (they involve *distances*).

4. In a topological space $(S, \mathcal{G})$, define

$$\mathcal{G}^0 = \mathcal{G},\ \mathcal{G}^1 = \mathcal{G}_\delta,\ \mathcal{G}^2 = \mathcal{G}_{\delta\sigma},\ \ldots$$

and

$$\mathcal{F}^0 = \mathcal{F},\ \mathcal{F}^1 = \mathcal{F}_\sigma,\ \mathcal{F}^2 = \mathcal{F}_{\sigma\delta},\ \mathcal{F}^3 = \mathcal{F}_{\sigma\delta\sigma},\ \text{etc.}$$

(Give an *inductive* definition.) Then prove by induction that

(a) $\mathcal{G}^n \subseteq \mathcal{B}$, $\mathcal{F}^n \subseteq \mathcal{B}$;

(b) $\mathcal{G}^{n-1} \subseteq \mathcal{G}^n$, $\mathcal{F}^{n-1} \subseteq \mathcal{F}^n$;

(c) $(\forall\, X \subseteq S)\ X \in \mathcal{F}^n$ iff $-X \in \mathcal{G}^n$;

(d) $(\forall\, X, Y \in \mathcal{F}^n)\ X \cap Y \in \mathcal{F}^n$, $X \cup Y \in \mathcal{F}^n$; same for $\mathcal{G}^n$;

(e) $(\forall\, X \in \mathcal{G}^n)\ (\forall\, Y \in \mathcal{F}^n)\ X - Y \in \mathcal{G}^n$ and $Y - X \in \mathcal{F}^n$.
[Hint: $X - Y = X \cap -Y$.]

5. For metric and pseudometric spaces (see Problem 2) prove that

$$\mathcal{F}^n \subseteq \mathcal{G}^{n+1} \text{ and } \mathcal{G}^n \subseteq \mathcal{F}^{n+1}$$

(cf. Problem 4).
[Hint for $\mathcal{F} \subseteq \mathcal{G}_\delta$: Let $F \in \mathcal{F}$. Set

$$G_n = \bigcup_{p \in F} G_p\left(\frac{1}{n}\right);$$

so

$$(\forall\, n) \quad F \subseteq G_n \in \mathcal{G}.$$

Hence

$$F \subseteq \bigcap_n G_n \in \mathcal{G}_\delta.$$

Also,

$$\bigcap_n G_n = \overline{F} = F$$

by Theorem 3 in Chapter 3, §16. Hence deduce that

$$(\forall\, F \in \mathcal{F}) \quad F \in \mathcal{G}_\delta,$$

so $\mathcal{F} \subseteq \mathcal{G}_\delta$; hence $\mathcal{G} \subseteq \mathcal{F}_\sigma$ by Problem 4(c). Now use induction.]

6. If m is as in Definition 5, then prove the following.

(i) m is regular.

(ii) $(\forall\, A \in \mathcal{M})\ mA = \sup\{mX \mid A \supseteq X \in \mathcal{M} \cap \mathcal{F}\}$.

(iii) The latter *implies* strong regularity if $m < \infty$ and $S \in \mathcal{M}$.

7. Let $\mu\colon \mathcal{B} \to E^*$ be a Borel measure in a metric space (S, ρ). Set

$$(\forall\, A \subseteq S) \quad n^* A = \inf\{\mu X \mid A \subseteq X \in \mathcal{G}\}.$$

Prove that

(i) n^* is an outer measure in S;

(ii) $n^* = \mu$ on $\mathcal{G}$;

(iii) the n^*-induced measure, $n\colon \mathcal{N}^* \to E^*$, is *topological* (so $\mathcal{B} \subseteq \mathcal{N}^*$);

(iv) $n \geq \mu$ on $\mathcal{B}$;

(v) $(\forall\, A \subseteq S)\ (\exists\, H \in \mathcal{G}_\delta)\ A \subseteq H$ and $\mu H = n^* A$.

[Hints: (iii) Using Problem 15 in §5 and Problem 12 in §6, let

$$\rho(X, Y) > \varepsilon > 0, \quad U = \bigcup_{x \in X} G_x\Big(\frac{1}{2}\varepsilon\Big), \quad V = \bigcup_{y \in Y} G_y\Big(\frac{1}{2}\varepsilon\Big).$$

Verify that $U, V \in \mathcal{G}$, $U \supseteq X$, $V \supseteq Y$, $U \cap V = \emptyset$.

By the definition of n^*,

$$(\exists\, G \in \mathcal{G}) \quad G \supseteq X \cup Y \text{ and } n^* G \leq n^*(X \cup Y) + \varepsilon;$$

also, $X \subseteq G \cap U$ and $Y \subseteq G \cap V$. Thus by (ii),

$$n^* X \leq \mu(G \cap U) \text{ and } n^* Y \leq \mu(G \cap V).$$

Hence

$$n^* X + n^* Y \leq \mu(G \cap U) + \mu(G \cap V) = \mu((G \cap U) \cup (G \cap V)) \leq \mu G = n^* G \leq n^*(X \cup Y) + \varepsilon.$$

Let $\varepsilon \to 0$ to get the CP: $n^* X + n^* Y \leq n^*(X \cup Y)$.

(iv) We have $(\forall\, A \in \mathcal{B})$

$$nA = n^* A = \inf\{\mu X \mid A \subseteq X \in \mathcal{G}\} \geq \inf\{\mu X \mid A \subseteq X \in \mathcal{B}\} = \mu A.$$

(Why?)

(v) Use the hint to Problem 11 in §5.]

8. From Problem 7 with $m = \mu$, prove that if

$$A \subseteq G \in \mathcal{G},$$

with $mG < \infty$ and $A \in \mathcal{B}$, then $mA = nA$.

[Hint: A, G, and $(G - A) \in \mathcal{B}$. By Problem 7(iii), $\mathcal{B} \subseteq N^*$ and n is additive on $\mathcal{B}$; so by Problem 7(ii)(iv),

$$nA = nG - n(G - A) \leq mG - m(G - A) = mA \leq nA.$$

Thus $mA = nA$. Explain all!]

9. Let m, n, and n^* be as in Problems 7 and 8. Suppose

$$S = \bigcup_{n=1}^{\infty} G_n,$$

with $G_n \in \mathcal{G}$ and $mG_n < \infty$ (this is called σ^0-*finiteness*).

Prove that

(i) $m = n$ on $\mathcal{B}$, and

(ii) m and n are strongly regular.

[Hints: Fix $A \in \mathcal{B}$. Show that

$$A = \bigcup A_n \ (\textit{disjoint})$$

for some Borel sets $A_n \subseteq G_n$ (use Corollary 1 in §1). By Problem 8, $mA_n = nA_n$ since

$$A_n \subseteq G_n \in \mathcal{G}$$

and $mG_n < \infty$. Now use σ-additivity to find $mA = nA$.

(ii) Use $\mathcal{G}$-regularity, part (i), and Theorem 1.]

10. Continuing Problems 8 and 9, show that n is the Lebesgue extension of m (see Theorem 2 in §6 and Note 3 in §6).

Thus *every* σ^0*-finite Borel measure* m *in* (S, ρ) *and its Lebesgue extension are strongly regular.*

[Hint: m induces an outer measure m^*, with $m^* = m$ on $\mathcal{B}$. It suffices to show that $m^* = n^*$ on 2^S. (Why?)

So let $A \subseteq S$. By Problem 7(v),

$$(\exists\, H \in \mathcal{B})\ A \subseteq H \text{ and } n^*A = mH = m^*H.$$

Also,

$$(\exists\, K \in \mathcal{B})\ A \subseteq K \text{ and } m^*A = mK$$

(Problem 12 in §5). Deduce that

$$n^*A \le n(H \cap K) = m(H \cap K) \le mH = n^*A$$

and

$$n^*A = m(H \cap K) = m^*A.]$$

§8. Lebesgue Measure

We shall now consider the most important example of a measure in E^n, due to Lebesgue. This measure generalizes the notion of *volume* and assigns "volumes" to a large set family, the "Lebesgue measurable" sets, so that "volume" becomes a complete topological measure. For "bodies" in E^3, this measure agrees with our intuitive idea of "volume."

We start with the volume function $v\colon \mathcal{C} \to E^1$ ("*Lebesgue premeasure*") on the semiring $\mathcal{C}$ of all intervals in E^n (§1). As we saw in §§5 and 6, this premeasure induces an outer measure m^* on all subsets of E^n; and m^*, in turn, induces a measure m on the σ-field $\mathcal{M}^*$ of m^*-measurable sets. These sets are, by definition, the *Lebesgue-measurable* (briefly *L-measurable*) sets; m^* and m so defined are the (n*-dimensional*) *Lebesgue outer measure* and *Lebesgue measure*.

Theorem 1. *Lebesgue premeasure v is σ-additive on $\mathcal{C}$, the intervals in E^n. Hence the latter are Lebesgue measurable ($\mathcal{C} \subseteq \mathcal{M}^*$), and the volume of each interval equals its Lebesgue measure:*

$$v = m^* = m \text{ on } \mathcal{C}.$$

This follows by Corollary 1 in §2 and Theorem 2 of §6.

Note 1. As $\mathcal{M}^*$ is a *σ-field* (§6), it is closed under countable unions, countable intersections, and differences. Thus

$$\mathcal{C} \subseteq \mathcal{M}^* \text{ implies } \mathcal{C}_\sigma \subseteq \mathcal{M}^*;$$

i.e., *any countable union of intervals is L-measurable.* Also, $E^n \in \mathcal{M}^*$.

Corollary 1. *Any countable set $A \subset E^n$ is L-measurable, with $mA = 0$.*

The proof is as in Corollary 6 of §2.

Corollary 2. *The Lebesgue measure of E^n is ∞.*

Prove as in Corollary 5 of §2.

Examples.

(a) Let

$$R = \{\text{rationals in } E^1\}.$$

Then R is *countable* (Corollary 3 of Chapter 1, §9); so $mR = 0$ by Corollary 1. Similarly for R^n (rational points in E^n).

(b) The measure of an interval with endpoints a, b in E^1 is its *length*, $b - a$. Let

$$R_o = \{\text{all rationals in } [a, b]\};$$

so $mR_o = 0$. As $[a, b]$ and R_o are in $\mathcal{M}^*$ (a σ-field), so is

$$[a, b] - R_o,$$

the *irrationals* in $[a, b]$. By Lemma 1 in §4, if $b > a$, then

$$m([a, b] - R_o) = m([a, b]) - mR_o = m([a, b]) = b - a > 0 = mR_o.$$

This shows again that the irrationals form a "larger" set than the rationals (cf. Theorem 3 of Chapter 1, §9).

(c) There are *uncountable* sets of measure zero (see Problems 8 and 10 below).

Theorem 2. *Lebesgue measure in E^n is complete, topological, and totally σ-finite. That is,*

(i) *all null sets (subsets of sets of measure zero) are L-measurable;*

(ii) *so are all open sets ($\mathcal{M}^* \supseteq \mathcal{G}$), hence all Borel sets ($\mathcal{M}^* \supseteq \mathcal{B}$); in particular, $\mathcal{M}^* \supseteq \mathcal{F}$, $\mathcal{M}^* \supseteq \mathcal{G}_\delta$, $\mathcal{M}^* \supseteq \mathcal{F}_\sigma$, $\mathcal{M}^* \supseteq \mathcal{F}_{\sigma\delta}$, etc.;*

(iii) *each* $A \in \mathcal{M}^*$ *is a countable union of disjoint sets of finite measure.*

Proof. (i) This follows by Theorem 1 in §6.

(ii) By Lemma 2 in §2, each open set is in $\mathcal{C}_\sigma$, hence in $\mathcal{M}^*$ (Note 1). Thus $\mathcal{M}^* \supseteq \mathcal{G}$. But by definition, the Borel field $\mathcal{B}$ is the *least* σ-ring $\supseteq \mathcal{G}$. Hence $\mathcal{M}^* \supseteq \mathcal{B}^*$.

(iii) As E^n is open, it is a countable union of disjoint half-open intervals,

$$E^n = \bigcup_{k=1}^{\infty} A_k \ (disjoint),$$

with $mA_k < \infty$ (Lemma 2 in §2). Hence

$$(\forall\, A \subseteq E^n) \quad A \subseteq \bigcup A_k;$$

so

$$A = \bigcup_k (A \cap A_k) \ (disjoint).$$

If, further, $A \in \mathcal{M}^*$, then $A \cap A_k \in \mathcal{M}^*$, and

$$m(A \cap A_k) \le mA_k < \infty. \text{ (Why?)} \quad \square$$

Note 2. More generally, a σ-finite set $A \in \mathcal{M}$ in a measure space $(S, \mathcal{M}, \mu)$ is a countable union of *disjoint* sets of finite measure (Corollary 1 of §1).

Note 3. *Not all* L-measurable sets are Borel sets. On the other hand, *not all* sets in E^n are L-measurable (see Problems 6 and 9 below.)

Theorem 3.

(a) *Lebesgue outer measure* m^* *in* E^n *is* $\mathcal{G}$*-regular; that is,*

(1) $$(\forall\, A \subseteq E^n) \quad m^*A = \inf\{mX \mid A \subseteq X \in \mathcal{G}\}$$

($\mathcal{G} =$ *open sets in* E^n).

(b) *Lebesgue measure* m *is strongly regular* (*Definition 5 and Theorems 1 and 2, all in* §7).

Proof. By definition, m^*A is the glb of all basic covering values of A. Thus given $\varepsilon > 0$, there is a basic covering $\{B_k\} \subseteq \mathcal{C}$ of nonempty sets B_k such that

(2) $$A \subseteq \bigcup B_k \text{ and } m^*A + \frac{1}{2}\varepsilon \ge \sum_k vB_k.$$

(Why? What if $m^*A = \infty$?)

Now, by Lemma 1 in §2, fix for each B_k an *open* interval $C_k \supseteq B_k$ such that

$$vC_k - \frac{\varepsilon}{2^{k+1}} < vB_k.$$

Then (2) yields

$$m^*A+\frac{1}{2}\varepsilon \geq \sum_k\Big(vC_k-\frac{\varepsilon}{2^{k+1}}\Big)=\sum_k vC_k-\frac{1}{2}\varepsilon;$$

so by σ-subadditivity,

$$m\bigcup_k C_k \leq \sum_k mC_k=\sum_k vC_k\leq m^*A+\varepsilon. \tag{3}$$

Let

$$X=\bigcup_k C_k.$$

Then X is open (as the C_k are). Also, $A\subseteq X$, and by (3),

$$mX\leq m^*A+\varepsilon.$$

Thus, indeed, m^*A is the *glb* of all mX, $A\subseteq X\in\mathcal{G}$, proving (a).

In particular, if $A\in\mathcal{M}^*$, (1) shows that m is regular (for $m^*A=mA$). Also, by Theorem 2, m is σ-finite, and $E^n\in\mathcal{M}^*$; so (b) follows by Theorem 1 in §7. □

Definition.

Given $A\subseteq E^n$ and $\bar{p}\in E^n$, let $\bar{p}+A$ or $A+\bar{p}$ denote the set of all points of the form

$$\bar{x}+\bar{p},\quad \bar{x}\in A.$$

We call $A+\bar{p}$ the *translate* of A by $\bar{p}$.

Theorem 4. *Lebesgue outer measure m^* and Lebesgue measure m in E^n are translation invariant. That is,*

(i) $(\forall\, A\subseteq E^n)\ (\forall\,\bar{p}\in E^n)\ m^*A=m^*(A+\bar{p})$;

(ii) *if A is L-measurable, so is $A+\bar{p}$, and $mA=m(A+\bar{p})$.*

See also Problem 7 in §10.

Proof. (i) If A is an interval with endpoints $\bar{a}$ and $\bar{b}$, then $A+\bar{p}$ is the interval with endpoints $\bar{a}+\bar{p}$ and $\bar{b}+\bar{p}$. (Verify!)

Hence the edge lengths of A and $A+\bar{p}$ are *the same*,

$$\ell_k=b_k-a_k=(b_k+p_k)-(a_k+p_k),\quad k=1,2,\dots,n.$$

Thus

$$mA=vA=\prod_{k=1}^{n}\ell_k=m(A+\bar{p});$$

so the theorem holds for *intervals*.

In the general case, m^*A is the glb of all basic covering values of A. But a basic covering consists of *intervals* that, when translated by $\bar{p}$, cover $A+\bar{p}$ and *retain the same volumes*, as was shown above.

Hence any covering *value* for A is also one for $A+\bar{p}$, and conversely (since A, in turn, is a translate of $A+\bar{p}$ by $-\bar{p}$).

Thus the basic covering values of A and of $A+\bar{p}$ are *the same*, with one and the same glb. Hence

$$m^*A = m^*(A+\bar{p}),$$

as claimed.

(ii) Now let $A \in \mathcal{M}^*$. We must show that

$$A+\bar{p} \in \mathcal{M}^*,$$

i.e., that

$$(\forall\, X \subseteq A+\bar{p})\ (\forall\, Y \subseteq -(A+\bar{p}))\quad m^*X + m^*Y = m^*(X \cup Y).$$

Thus fix $X \subseteq A+\bar{p}$ and $Y \subseteq -(A+\bar{p})$.

As is easily seen, $X-\bar{p} \subseteq A$ and $Y-\bar{p} \subseteq -A$ (translate all by $-\bar{p}$). Since $A \in \mathcal{M}^*$, we get

$$m^*(X-\bar{p}) + m^*(Y-\bar{p}) = m^*((X \cup Y)-\bar{p}).$$

(Why?) But by (i), $m^*X = m^*(X-\bar{p})$, $m^*Y = m^*(Y-\bar{p})$, and

$$m^*(X \cup Y) = m^*((X \cup Y)-\bar{p}).$$

Hence

$$m^*X + m^*Y = m^*(X \cup Y),$$

and so $A+\bar{p} \in \mathcal{M}^*$.

Now, as $m^* = m$ on $\mathcal{M}^*$, (i) yields $mA = m(A+\bar{p})$, proving (ii) also. □

Problems on Lebesgue Measure

1. Fill in all details in the proof of Theorems 3 and 4.

1′. Prove Note 2.

2. From Theorem 3 deduce that

$$(\forall\, A \subseteq E^n)\ (\exists\, B \in \mathcal{G}_\delta)\quad A \subseteq B \text{ and } m^*A = mB.$$

[Hint: See the hint to Problem 7 in §5.]

3. Review Problem 3 in §5.

4. Consider all translates

$$R+p \quad (p \in E^1)$$

of

$$R = \{\text{rationals in } E^1\}.$$

Prove the following.

(i) Any two such translates are either disjoint or identical.

(ii) Each $R + p$ contains at least one element of $[0, 1]$.

[Hint for (ii): Fix a *rational* $y \in (-p, 1 - p)$, so $0 < y + p < 1$. Then $y + p \in R + p$, and $y + p \in [0, 1]$.]

5. Continuing Problem 4, choose *one* element $q \in [0, 1]$ from *each* $R + p$. Let Q be the set of all q so chosen.

Call a translate of Q, $Q + r$, "good" iff $r \in R$ and $|r| < 1$. Let U be the union of all "good" translates of Q.

Prove the following.

(a) There are only countably many "good" $Q + r$.

(b) All of them lie in $[-1, 2]$.

(c) Any two of them are either disjoint or identical.

(d) $[0, 1] \subseteq U \subseteq [-1, 2]$; hence $1 \leq m^*U \leq 3$.

[Hint for (c): Suppose

$$y \in (Q + r) \cap (Q + r').$$

Then

$$y = q + r = q' + r' \quad (q, q' \in Q,\ r, r' \in R);$$

so $q = q' + (r' - r)$, with $(r' - r) \in R$.

Thus $q \in R + q'$ and $q' = 0 + q' \in R + q'$. Deduce that $q = q'$ and $r = r'$; hence $Q + r = Q + r'$.]

6. Show that Q in Problem 5 is *not* L-measurable.

[Hint: Otherwise, by Theorem 4, each $Q + r$ is L-measurable, with $m(Q + r) = mQ$. By 5(a)(c), U is a *countable disjoint* union of "good" translates.

Deduce that $mU = 0$ if $mQ = 0$, or $mU = \infty$, *contrary to* 5(d).]

7. Show that if $f\colon S \to T$ is continuous, then $f^{-1}[X]$ is a Borel set in S whenever $X \in \mathcal{B}$ in T.

[Hint: Using Note 1 in §7, show that

$$\mathcal{R} = \{X \subseteq T \mid f^{-1}[X] \in \mathcal{B} \text{ in } S\}$$

is a σ-ring in T. As $\mathcal{B}$ is the *least* σ-ring $\supseteq \mathcal{G}$, $\mathcal{R} \supseteq \mathcal{B}$ (the Borel field *in* T.]

8. Prove that every degenerate interval in E^n has Lebesgue measure 0, *even if it is uncountable*. Give an example in E^2. *Prove* uncountability.

[Hint: Take $\bar{a} = (0, 0)$, $\bar{b} = (0, 1)$. Define $f\colon E^1 \to E^2$ by $f(x) = (0, x)$. Show that f is one-to-one and that $[\bar{a}, \bar{b}]$ is the f-image of $[0, 1]$. Use Problem 2 of Chapter 1, §9.]

9. Show that *not all* L-measurable sets are Borel sets in E^n.

[Hint for E^2: With $[\bar{a}, \bar{b}]$ and f as in Problem 8, show that f is continuous (use the sequential criterion). As $m[\bar{a}, \bar{b}] = 0$, all subsets of $[\bar{a}, \bar{b}]$ are in $\mathcal{M}^*$ (Theorem 2(i)), hence in $\mathcal{B}$ if we assume $\mathcal{M}^* = \mathcal{B}$. But then by Problem 7, the same would apply to subsets of $[0, 1]$, *contrary to* Problem 6.

Give a similar proof for E^n $(n > 1)$.

Note: In E^1, too, $\mathcal{B} \neq \mathcal{M}^*$, but a different proof is necessary. We omit it.]

10. Show that Cantor's set P (Problem 17 in Chapter 3, §14) has Lebesgue measure zero, even though it is *uncountable*.

[Outline: Let

$$U = [0, 1] - P;$$

so U is the union of open intervals *removed* from $[0, 1]$. Show that

$$mU = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 1$$

and use Lemma 1 in §4.]

11. Let $\mu\colon \mathcal{B} \to E^*$ be the *Borel restriction* of Lebesgue measure m in E^n (§7). Prove that

(i) μ in incomplete;

(ii) m is the Lebesgue extension (*and *completion*, as in Problem 15 of §6) of μ.

[Hints: (i) By Problem 9, some μ-null sets are *not* in $\mathcal{B}$. (ii) See the proof (end) of Theorem 2 in §9 (the *next* section).]

12. Prove the following.

(i) All intervals in E^n are Borel sets.

(ii) The σ-ring generated by *any* one of the families $\mathcal{C}$ or $\mathcal{C}'$ in Problem 3 of §5 coincides with the Borel field in E^n.

[Hints: (i) Any interval arises from a *closed* one by dropping some "faces" (degenerate *closed* intervals). (ii) Use Lemma 2 from §2 and Problem 7 of §3.]

***13.** Show that if a measure $m'\colon \mathcal{M}' \to E^*$ in E^n agrees on *intervals* with Lebesgue measure $m\colon \mathcal{M}^* \to E^*$, then the following are true.

(i) $m' = m$ on $\mathcal{B}$, the Borel field in E^n.

(ii) If m' is also complete, then $m' = m$ on $\mathcal{M}^*$.

[Hint: (i) Use Problem 13 of §5 and Problem 12 above.]

14. Show that globes of equal radius have the same Lebesgue measure.

[Hint: Use Theorem 4.]

15. Let $f\colon E^n \to E^n$, with

$$f(\bar{x}) = c\bar{x} \quad (0 < c < \infty).$$

Prove the following.

(i) $(\forall\, A \subseteq E^n)\ m^* f[A] = c^n m^* A$ (m^* = Lebesgue outer measure).

(ii) $A \in \mathcal{M}^*$ iff $f[A] \in \mathcal{M}^*$.

[Hint: If, say, $A = (\bar{a}, \bar{b}]$, then $f[A] = (c\bar{a}, c\bar{b}]$. (Why?) Proceed as in Theorem 4, using f^{-1} also.]

16. From Problems 14 and 15 show that

(i) $mG_{\bar{p}}(cr) = c^n \cdot mG_{\bar{p}}(r)$;

(ii) $mG_{\bar{p}}(r) = m\overline{G}_{\bar{p}}(r)$;

(iii) $mG_{\bar{p}}(r) = a \cdot mI$, where I is the cube inscribed in $G_{\bar{p}}(r)$ and

$$a = \left(\frac{1}{2}\sqrt{n}\right)^n \cdot mG_{\bar{0}}(1).$$

[Hints: (i) $f[G_{\bar{0}}(r)] = G_{\bar{0}}(cr)$. (ii) Prove that

$$mG_{\bar{p}} \le m\overline{G}_{\bar{p}} \le c^n mG_{\bar{p}}$$

if $c > 1$. Let $c \to 1$.]

17. Given $a < b$ in E^1, let $\{r_n\}$ be the sequence of all *rationals* in $A = [a, b]$. Set $(\forall\, n)$

$$\delta_n = \frac{b-a}{2^{n+1}}$$

and

$$G_n = (a_n, b_n) = (a, b) \cap \left(r_n - \frac{1}{2}\delta_n,\ r_n + \frac{1}{2}\delta_n\right).$$

Let

$$P = A - \bigcup_{n=1}^{\infty} G_n.$$

Prove the following.

(i) $\sum_{n=1}^{\infty} \delta_n = \frac{1}{2}(b-a) = \frac{1}{2} mA$.

(ii) P is closed; $P^o = \emptyset$, yet $mP > 0$.

(iii) The G_n can be made *disjoint* (see Problem 3 in §2), with mP still > 0.

(iv) Construct such a $P \subseteq A$ ($P = \overline{P}$, $P^o = \emptyset$) of *prescribed* measure $mP = \varepsilon > 0$.

18. Find an open set $G \subset E^1$, with $mG < m\overline{G} < \infty$.
[Hint: $G = \bigcup_{n=1}^{\infty} G_n$ with G_n as in Problem 17.]

***19.** If $A \subseteq E^n$ is open and *convex*, then $mA = m\overline{A}$.
[Hint: Let first $\bar{0} \in A$. Argue as in Problem 16.]

§9. Lebesgue–Stieltjes Measures

Let

$$\alpha\colon E^1 \to E^1$$

be a *nondecreasing* function ($\alpha\uparrow$). Consider the *Lebesgue–Stieltjes set function* s_α (Example (d) in §4).

As we noted in Problem 7 of §4, $s_\alpha \geq 0$ when $\alpha\uparrow$; for then

$$s_\alpha(a,b) = \alpha(b-) - \alpha(a+) \geq 0.$$

Similarly for other intervals. Also, $\emptyset \in \mathcal{C}$ and $s_\alpha\emptyset = 0$ by definition.

Thus s_α is a *premeasure* on $\mathcal{C}$ (finite intervals in E^1), called the *α-induced Lebesgue–Stieltjes (LS) premeasure* in E^1.

The outer measure m_α^* induced by s_α (§5) is called the *α-induced LS outer measure*; its restriction to the family $\mathcal{M}_\alpha^*$ of m_α^*-measurable (or *LS-measurable*) sets is the *α-induced LS measure on* E^1, denoted m_α.

Recall that, by our definitions, premeasures, outer measures, and measures are all *nonnegative*.

Note 1. No generality is lost by assuming that α is *right continuous* (if not, replace it by the right-continuous function $\beta\uparrow$, with $\beta(x) = \alpha(x+)$). Similarly, one achieves *left* continuity by setting $\beta(x) = \alpha(x-)$.

Note 2. If α is right continuous, one often restricts s_α to the family $\mathcal{C}^*$ of all *half-open* intervals (for motivation, see Problem 7(iv) in §4). This does not affect m_α^* or m_α (Problem 3′ in §5), and simplifies the proof of additivity

$$s_\alpha(a,b] + s_\alpha(b,c] = \alpha(b) - \alpha(a) + \alpha(c) - \alpha(b) = \alpha(c) - \alpha(a) = s_\alpha(a,c].$$

Recall that both $\mathcal{C}$ and $\mathcal{C}^*$ are *semirings* (Note 1 in §1).

Theorem 1. *The LS premeasure s_α is σ-additive on the semiring $\mathcal{C}$ of all finite intervals in E^1.*

Hence (by Theorem 2 in §6) all such intervals are LS-measurable ($\mathcal{C} \subseteq \mathcal{M}_\alpha^*$), and

$$m_\alpha A = s_\alpha A$$

for any such interval A.

Proof. As is easily seen, s_α is additive (Problem 7 of §4).

It also satisfies Lemma 1 of §1 and Lemma 1 in §2 (Problem 7(v) in §4).

The proof of σ-additivity is then quite analogous to that of Theorem 1 of §2; we omit its repetition.

The rest is immediate by Theorem 2 of §6. □

Similarly, the proofs of Theorems 2 and 3 (*but not* 4) of §8 carry over to LS measures. Thus *LS measures are complete*, *topological*, *totally σ-finite and strongly regular*.

As in §8, it follows that singletons and countable sets are measurable, but *their LS measure need not be* 0 (Problem 8(iii) in §4).

Also, $E^1 \in \mathcal{M}^*_\alpha$, but $m_\alpha E^1$ may be *finite* (Problem 8(ii)(ii′) in §4).

Since the proofs are the same as in §8, we omit them.

Note, however, the following facts.

(i) For *singletons*, $m_\alpha\{p\} = 0$ iff α is continuous at p (Problem 7(ii) in §4).

(ii) Hence

$$m_\alpha[a,b] = m_\alpha(a,b] = m_\alpha[a,b) = m_\alpha(a,b) = \alpha(b) - \alpha(a)$$

iff α is continuous at a and b (Problem 7(iv) in §4).

(iii) LS measures need not be translation invariant (Problem 8(i) of §4).

(iv) If $\alpha(x) = x$ on E^1, then $m^*_\alpha = m^*$ (= Lebesgue outer measure in E^1).

Thus *Lebesgue measure is a special case of LS measure.*

The latter is a kind of "weighted length." Imagine that mass is distributed along the line, with $\alpha(x)$ equal to the mass of

$$(-\infty, x].$$

For simplicity, assume that α is right-continuous (cf. Notes 1 and 2). Then the mass of $(a,b]$ is

$$\alpha(b) - \alpha(a),$$

and p is a "point mass" iff

$$m_\alpha\{p\} > 0.$$

Our next theorem shows that LS measures practically exhaust all topological measures in E^1 of any importance. We shall use Notes 1 and 2 above.

***Theorem 2.** *Let $m\colon \mathcal{M} \to E^*$ be a topological measure in E^1, finite on $\mathcal{C}^*$ (half-open intervals). Then there is an LS measure m_α such that*

$$m_\alpha = m$$

on the Borel field $\mathcal{B}$ in E^1.

If m is also complete, then $m_\alpha = m$ on all of $\mathcal{M}^*_\alpha$.

Proof. Define α as follows:

$$\alpha(x) = \begin{cases} m(0,x] & \text{if } x \geq 0, \\ -m(x,0] & \text{if } x < 0. \end{cases}$$

Clearly, $\alpha\uparrow$ on E^1. Also, the right continuity of m (Theorem 2 of §4) implies that of α. (Verify!)

Thus α induces an LS measure m_α, with

$$m_\alpha(a,b] = s_\alpha(a,b] = \alpha(b) - \alpha(a)$$

(Problem 7(iv) in §4). We claim that $m_\alpha = m$ *on* $\mathcal{B}$.

First, consider any $(a,b] \in \mathcal{C}^*$. If $0 \le a \le b$, then

$$m(a,b] = m(0,b] - m(0,a] = \alpha(b) - \alpha(a) = m_\alpha(a,b].$$

Similarly in the cases $a < 0 \le b$ and $a \le b < 0$. Thus

$$m_\alpha = m \text{ (\textit{finite}) on } \mathcal{C}^*.$$

By Problem 13 in §5,

$$m_\alpha = m \textit{ on } \mathcal{B},$$

the σ-ring generated by $\mathcal{C}^*$ (Problem 12 of §8). Thus m and m_α have *the same restriction to* $\mathcal{B}$ (call it μ).

Now, by Note 3 in §6, μ induces an outer measure μ^*.

As $\mathcal{B} \supseteq \mathcal{C}_\sigma^*$, both μ^* and m_α^* are $\mathcal{B}$*-regular*, by Theorem 3 in §5. Thus

$$(\forall\, A \subseteq E^1) \quad m_\alpha^*(A) = \inf\{\mu X \mid A \subseteq X \in B\} = \mu^* A,$$

i.e., $m_\alpha^* = \mu^*$, and so m_α is the restriction of *both* m_α^* and μ^* to measurable sets. Hence m_α is the *Lebesgue extension* of μ, by definition.

By Problem 17 in §6, $m_\alpha = \overline{\mu}$ is the "*least*" complete extension of μ. Thus if m is complete, it is an *extension* of m_α; so $m = m_\alpha$ on $\mathcal{M}_\alpha^*$, as claimed. □

Problems on Lebesgue–Stieltjes Measures

1. Do Problems 7 and 8 in §4 and Problem 3′ in §5, if not done before.

2. Prove in detail Theorems 1 to 3 in §8 for LS measures and outer measures.

3. Do Problem 2 in §8 for LS-outer measures in E^1.

4. Prove that $f\colon E^1 \to (S,\rho)$ is right (left) continuous at p iff

$$\lim_{n\to\infty} f(x_n) = f(p) \text{ as } x_n \searrow p \ (x_n \nearrow p).$$

[Hint: Modify the proof of Theorem 1 in Chapter 4, §2.]

5. Fill in all proof details in Theorem 2.
[Hint: Use Problem 4.]

6. In Problem 8(iv) of §4, describe m_α^* and M_α^*.

7. Show that if $\alpha = c$ (constant) on an *open* interval $I \subseteq E^1$ then

$$(\forall\, A \subseteq I) \quad m_\alpha^*(A) = 0.$$

Disprove it for *nonopen* intervals I (give a counterexample).

8. Let $m'\colon \mathcal{M} \to E^*$ be a *topological, translation-invariant measure* in E^1, with $m'(0,1] = c < \infty$. Prove the following.

(i) $m' = cm$ on the Borel field $\mathcal{B}$. (Here $m\colon \mathcal{M}^* \to E^*$ is *Lebesgue* measure in E^1.)

*(ii) If m' is also complete, then $m' = cm$ on $\mathcal{M}^*$.

(iii) If $0 < c < \infty$, some set $Q \subset [0,1]$ is *not* m'-measurable.

*(iv) If $\mathcal{M}' = \mathcal{B}$, then cm is the *completion* of m' (Problem 15 in §6).

[Outline: (i) By additivity and translation invariance,

$$m'(0,r] = cm(0,r]$$

for *rational*

$$r = \frac{n}{k}, \quad n, k \in N$$

(first take $r = n$, then $r = \frac{1}{k}$, then $r = \frac{n}{k}$).

By right continuity (Theorem 2 in §4), prove it for *real* $r > 0$ (take rationals $r_i \searrow r$).

By translation, $m' = cm$ on half-open intervals. Proceed as in Problem 13 of §8.

(iii) See Problems 4 to 6 in §8. Note that, by Theorem 2, one may assume $m' = m_\alpha$ (a translation-invariant *LS measure*). As $m_\alpha = cm$ on half-open intervals, Lemma 2 in §2 yields $m_\alpha = cm$ on $\mathcal{G}$ (open sets). Use $\mathcal{G}$-regularity to prove $m^*_\alpha = cm^*$ and $\mathcal{M}^*_\alpha = \mathcal{M}^*$.]

***9.** (LS measures in E^n.) Let

$$\mathcal{C}^* = \{\text{half-open intervals in } E^n\}.$$

For any map $G\colon E^n \to E^1$ and any $(\bar{a}, \bar{b}] \in \mathcal{C}^*$, set

$$\begin{aligned} \Delta_k G(\bar{a}, \bar{b}] &= G(x_1, \ldots, x_{k-1}, b_k, x_{k+1}, \ldots, x_n) \\ &\quad - G(x_1, \ldots, x_{k-1}, a_k, x_{k+1}, \ldots, x_n), \quad 1 \le k \le n. \end{aligned}$$

Given $\alpha\colon E^n \to E^1$, set

$$s_\alpha(\bar{a}, \bar{b}] = \Delta_1(\Delta_2(\cdots(\Delta_n \alpha(\bar{a}, \bar{b}])\cdots)).$$

For example, in E^2,

$$s_\alpha(a, b] = \alpha(b_1, b_2) - \alpha(b_1, a_2) - [\alpha(a_1, b_2) - \alpha(a_1, a_2)].$$

Show that s_α is additive on $\mathcal{C}^*$. Check that the order in which the Δ_k are applied is immaterial. Set up a formula for s_α *in* E^3.
[Hint: First take *two* disjoint intervals

$$(\bar{a}, \bar{q}] \cup (\bar{p}, \bar{b}] = (\bar{a}, \bar{b}],$$

as in Figure 2 in Chapter 3, §7. Then use induction, as in Problem 9 of Chapter 3, §7.]

***10.** If s_α in Problem 9 is nonnegative, and α is right continuous in each variable x_k *separately*, we call α a *distribution function*, and s_α is called the *α-induced LS premeasure* in E^n; the *LS outer measure* m^*_α *and measure*

$$m_\alpha \colon \mathcal{M}^*_\alpha \to E^*$$

in E^n (obtained from s_α as shown in §§5 and 6) are said to be *induced by* α.

For s_α, m^*_α, and m_α so defined, redo Problems 1–3 above.

*§10. Vitali Coverings

Lebesgue measure m leads to an interesting analogue of the Heine–Borel theorem. Below, m^* is Lebesgue outer measure in E^n. We start with some definitions.

Definition 1.

A sequence $\{I_k\}$ of sets in a metric space (S, ρ) *converges to a point* p $(I_k \to p)$ iff

$$p \in \bigcap_{k=1}^{\infty} I_k$$

and

$$\lim_{k\to\infty} dI_k = 0,$$

where dI_k = diameter of I_k.

Definition 2.

A family $\mathcal{K}$ of nonempty sets in (S, ρ) is a *Vitali covering* (*V-covering*) of a set $A \subseteq (S, \rho)$ iff for each $p \in A$ there is a sequence $\{I_k\} \subseteq \mathcal{K}$, with $I_k \to p$.

We then also say that $\mathcal{K}$ covers A in the *Vitali sense* (*V-sense*).

Theorem 1 (Vitali)**.** *If a set $\mathcal{K}$ of nondegenerate cubes (or globes) in E^n covers A in the V-sense, then*

$$m^*(A - \bigcup_k I_k) = 0$$

for some disjoint *sequence* $\{I_k\} \subseteq \mathcal{K}$.

Proof. We give the proof for *cubes* (it is similar for *globes*).

First, suppose $A \subseteq I^o$ for some open cube I^o. Then A is also covered in the V-sense by the subfamily $\mathcal{K}^o \subseteq \mathcal{K}$ of *those cubes that lie in* I^o. (Why?) We also assume that $A \not\subseteq \bigcup I_j$ for any disjoint *finite* sequence $\{I_j\} \subseteq \mathcal{K}$ (otherwise, all is trivial). Finally , we assume that all cubes in $\mathcal{K}$ are *closed*; for other kinds of cubes, the theorem then easily follows (see Problem 3 below).

We claim that

$$(1) \qquad (\forall \textit{ disjoint cubes } I_1, \ldots, I_h \in \mathcal{K}^o)\ (\exists I \in \mathcal{K}^o) \quad I \cap \bigcup_{j=1}^{h} I_j = \emptyset.$$

Indeed, as

$$A \not\subseteq \bigcup_{j=1}^{h} I_j,$$

there is some

$$\bar{p} \in A - \bigcup_{j=1}^{h} I_j.$$

By assumption, all I_j are *closed*; so

$$-\bigcup_{j=1}^{h} I_j$$

is open. Hence there is a globe

$$G_{\bar{p}}(\delta) \subseteq -\bigcup_{j=1}^{h} I_j.$$

As $\mathcal{K}^o$ is a V-covering, it contains a sequence $I_i \to \bar{p}$, $dI_i \to 0$; so there is $I = I_i \in \mathcal{K}^o$ with $\bar{p} \in I$ and $dI < \delta$. Therefore,

$$I \subseteq G_{\bar{p}}(\delta) \subseteq -\bigcup_{j=1}^{h} I_j;$$

so

$$I \cap \bigcup_{j=1}^{h} I_j = \emptyset,$$

as claimed.

Now, using induction, suppose we have already fixed k disjoint cubes I_j in $\mathcal{K}^o$. By (1), there are cubes $I \in \mathcal{K}^o$ with

$$I \cap \bigcup_{j=1}^{k} I_j = \emptyset.$$

Let δ_k be the lub of their diameters. As all $I \in \mathcal{K}^o$ lie in I^o,

$$\delta_k = \sup\left\{ dI \;\middle|\; I \in \mathcal{K}^o,\ I \subseteq -\bigcup_{j=1}^{k} I_j \right\} \leq dI^o < \infty.$$

Hence by properties of the lub, we find a cube $I_{k+1} \in \mathcal{K}^o$ such that

$$I_k \subseteq -\bigcup_{j=1}^{k} I_j$$

and $dI_{k+1} > \frac{1}{2}\delta_k$.

In this way, taking $k = 1, 2, \ldots,$ we select a *disjoint* sequence $\{I_k\} \subseteq \mathcal{K}^o$ with $dI_{k+1} > \frac{1}{2}\delta_k$ for all k. We shall show that

$$m^*\left(A - \bigcup_{k=1}^{\infty} I_k \right) = 0$$

in four steps.

(I) Let ℓ_k be the edge length of I_k; so $dI_k = \ell_k\sqrt{n}$. (Why?)

Enclose each I_k in a cube J_k with the same center and with edge length

$$(4n+1)\,\ell_k.$$

Then

$$(\forall\, \bar{x} \in I_k)\ (\forall\, \bar{y} \notin J_k)$$

$$\rho(\bar{x}, \bar{y}) > 2n\,\ell_k \geq 2\ell_k\sqrt{n} \tag{2}$$
$$= 2\,dI_k > \delta_{k-1}.$$

(See Figure 32, where $n = 2$.) Also,

$$mJ_k = (4n+1)^n\, mI_k.$$

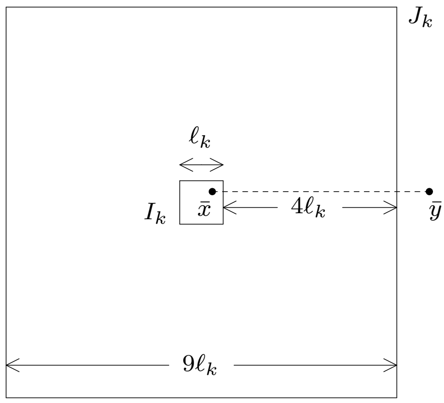

FIGURE 32

(II) As the I_k lie in I^o, the σ-additivity of m yields

$$\sum_{k=1}^{\infty} mJ_k = (4n+1)^n \sum_{k=1}^{\infty} mI_k$$
$$= (4n+1)^n\, m \bigcup_{k=1}^{\infty} I_k$$
$$\leq (4n+1)^n\, mI^o < \infty.$$

Thus the series $\sum mJ_k$ *converges*; so its "remainder" tends to 0:

$$\lim_{r\to\infty} \sum_{k=r}^{\infty} mJ_k = 0.$$

Also, $mJ_k \to 0$. But by definition,

$$\delta_k < 2dI_{k+1} < 2dJ_{k+1} = 2\sqrt{n}\,(mJ_{k+1})^{1/n} \quad (n \text{ fixed}).$$

Hence $\lim_{k\to\infty} \delta_k = 0$, too.

(III) Now, seeking a contradiction, suppose

$$m^*\left(A - \bigcup_{k=1}^{\infty} I_k\right) > 0.$$

Then as

$$\lim_{r\to\infty} \sum_{k=r}^{\infty} mJ_k = 0,$$

there is r such that

$$m \bigcup_{k=r}^{\infty} J_k \le \sum_{k=r}^{\infty} mJ_k < m^*\left(A - \bigcup_{k=1}^{\infty} I_k\right).$$

Hence

$$A - \bigcup_{k=1}^{\infty} I_k \not\subseteq \bigcup_{k=r}^{\infty} J_k.$$

(Why?) Thus there is

$$\bar{p} \in A - \bigcup_{k=1}^{\infty} I_k$$

not in

$$\bigcup_{k=r}^{\infty} J_k,$$

so that

$$(3) \qquad (\forall\, k \ge r) \quad \bar{p} \notin J_k,\ \bar{p} \in A, \text{ and } \bar{p} \in -\bigcup_{k=1}^{\infty} I_k \subseteq -\bigcup_{k=r}^{\infty} I_k.$$

As

$$-\bigcup_{k=1}^{r} I_k \in \mathcal{G},$$

we find (as before) a cube $K \in \mathcal{K}^o$ such that $\bar{p} \in K$ and

$$K \cap \bigcup_{k=1}^{r} I_k = \emptyset.$$

Also, as $\delta_k \to 0$, we have $\delta_k < dK$ for large k. But by our choice of the δ_k, this implies

$$K \cap \bigcup_{j=1}^{k} I_j \neq \emptyset$$

for large k (why?), whereas

$$K \cap \bigcup_{j=1}^{r} I_j = \emptyset,$$

as shown above.

Thus there is a *least* $k > r$, call it q, such that

$$K \cap I_q \neq \emptyset,$$

and $\delta_q < dK \leq \delta_{q-1}$.

By (3), $\bar{p} \notin J_q$. As

$$K \cap I_q \neq \emptyset,$$

let $\bar{x} \in K \cap I_q$. Since $\bar{x}, \bar{p} \in K$,

$$\rho(\bar{x}, \bar{p}) \leq dK < \delta_{q-1}.$$

But as $\bar{x} \in I_q$ and $\bar{p} \notin J_k$, we have

$$\rho(\bar{x}, \bar{p}) > \delta_{q-1}$$

by (2).

This contradiction proves the theorem for *bounded* sets A.

(IV) If A is not bounded, use Lemma 2 in §2 to find a sequence $\{K_i\}$ of *disjoint* half-open intervals with

$$\bigcup K_i = E^n \supseteq A.$$

Let

$$A_i = A \cap K_i^o,$$

where K_i^o is the *open* interval with the same endpoints; so $mK_i = mK_i^o$ and $m(K_i - K_i^o) = 0$.

Set

$$Z = \bigcup_{i=1}^{\infty} (K_i - K_i^o);$$

so $mZ = 0$ and

$$\bigcup_{i=1}^{\infty} K_i^o = E^n - Z.$$

(Why?) As $A_i = A \cap K_i^o$, we have

$$\bigcup_{i=1}^{\infty} A_i = A \cap \bigcup_{i=1}^{\infty} K_i^o = A \cap (E^n - Z) = A - Z. \tag{4}$$

Clearly, each A_i is covered in the V-sense by those $\mathcal{K}$-cubes that *lie in* K_i^o. Thus as shown above,

$$(\forall\, i) \quad m^*\left(A - \bigcup_j I_{ij}\right) = 0$$

for *disjoint* cubes $I_{ij} \subseteq K_i^o$. That is,

$$(\forall\, i) \quad \bigcup_j I_{ij} \cup Z_i \supseteq A_i,$$

where

$$Z_i = A_i - \bigcup_j I_{ij}$$

and $mZ_i = 0$. Hence by (4),

$$\bigcup_{i=1}^{\infty} \bigcup_j I_{ij} \cup \bigcup_i Z_i \supseteq \bigcup_i A_i = A - Z,$$

so that

$$m^*\left(A - \bigcup_{i,j} I_{ij}\right) = 0.$$

Rearranging the I_{ij} in a *single* sequence $\{I_k\}$, we complete the proof. □

Theorem 2. *If* $m^*A < \infty$ *in Theorem* 1, *then for every* $\varepsilon > 0$ *there is a* finite *disjoint sequence* $\{I_k\} \subseteq \mathcal{K}$ *such that*

$$m^*\left(A - \bigcup_k I_k\right) < \varepsilon.$$

Proof. Fix $\varepsilon > 0$. As $m^*A < \infty$, the $\mathcal{G}$-regularity of m^* (Theorem 3 of §8) yields an *open* $G \supseteq A$ such that

$$mG < m^*A + \varepsilon.$$

Clearly, A is covered in the V-sense by the subfamily $\mathcal{K}^o$ of those $\mathcal{K}$-sets that lie in G. Thus by Theorem 1,

$$m^*\left(A - \bigcup I_k\right) = 0$$

for a *disjoint* sequence $\{I_k\} \subseteq \mathcal{K}^o$. Also,

$$\bigcup I_k \subseteq G,$$

and so

$$\sum mI_k = m \bigcup I_k \leq mG < \infty.$$

Thus $\sum mI_k$ *converges*; so

$$\sum_{k=r}^{\infty} mI_k < \varepsilon$$

for large r.

On the other hand,

$$A - \bigcup_{k=1}^{r} I_k \subseteq \left(A - \bigcup_{k=1}^{\infty} I_k\right) \cup \bigcup_{k=r}^{\infty} I_k.$$

Hence

$$m^*\left(A - \bigcup_{k=1}^{r} I_k\right) \leq m^*\left(A - \bigcup_{k=1}^{\infty} I_k\right) + m^* \bigcup_{k=r}^{\infty} I_k \leq 0 + \sum_{k=r}^{\infty} mI_k < \varepsilon,$$

as required. □

As an application, we obtain the following important theorem.

Theorem 3 (Lebesgue). *If* $f\colon E^1 \to E^1$ *is monotone, it is differentiable* almost everywhere ("*a.e.*"), *i.e., on* $E^1 - Z$ *for some* Z *of Lebesgue measure zero.*

We sketch the proof in a few steps (lemmas). These lemmas anticipate a more general approach to be taken in §12, with the notation in the following definition.

Definition 3.

Let $m =$ Lebesgue measure and

$$\overline{\mathcal{K}} = \{\textit{all} \text{ cubes } I \subset E^n \text{ with } mI > 0\}.$$

Let

$$s\colon \mathcal{M}' \to [0, \infty], \quad \mathcal{M}' \supseteq \overline{\mathcal{K}},$$

be another measure in E^n, finite on $\overline{\mathcal{K}}$.

For any natural $r > 0$, and $\bar{p} \in E^n$, we set

$$g_r(\bar{p}) = \inf\left\{ \frac{sI}{mI} \,\middle|\, \bar{p} \in I \in \overline{\mathcal{K}},\ dI < \frac{1}{r} \right\}$$

and

$$h_r(\bar{p}) = \sup\left\{ \frac{sI}{mI} \;\middle|\; \bar{p} \in I \in \overline{\mathcal{K}},\ dI < \frac{1}{r} \right\};$$

furthermore, we denote

$$\underline{D}s(\bar{p}) = \sup_r g_r(\bar{p}) \text{ and } \overline{D}s(\bar{p}) = \inf_r h_r(\bar{p}).$$

Clearly, $\{g_r\}\uparrow$, $\{h_r\}\downarrow$, and

$$0 \leq \underline{D}s = \lim_{r\to\infty} g_r \leq \lim_{r\to\infty} h_r = \overline{D}s$$

at each $\bar{p} \in E^n$. (Why?)

We also write $J(\overline{D}s > i)$ for

$$\{\bar{x} \in J \mid \overline{D}s(\bar{x}) > i\},$$

$J(\underline{D}s = a)$ for

$$\{\bar{x} \in J \mid \underline{D}s(\bar{x}) = a\},$$

etc.

Lemma 1. *With the above notation,* $0 \leq \underline{D}s \leq \overline{D}s < \infty$ *a.e. on* E^n*.*

Proof Outline. Fix any *open* set $J \subset E^n$, with $mJ < \infty$ and $sJ < \infty$ (e.g., an open cube in $\overline{\mathcal{K}}$).

For $i = 1, 2, \ldots$ set

$$A_i = J(\overline{D}s > i)$$

and

$$\mathcal{K}_i = \left\{ I \in \overline{\mathcal{K}} \;\middle|\; I \subseteq J,\ \frac{sI}{mI} > i \right\}.$$

Verify that $\mathcal{K}_i$ is a V-covering of A_i; so there is a *disjoint* sequence $\{I_k\} \subseteq \mathcal{K}_i$, with

$$m^*\left(A_i - \bigcup I_k\right) = 0$$

and

$$\bigcup I_k \subseteq J.$$

Hence (cf. Problem 3 below)

$$m^*A_i \leq m\bigcup I_k = \sum mI_k \leq \frac{1}{i}\sum sI_k = \frac{1}{i}s\bigcup I_k \leq \frac{sJ}{i}, \quad i = 1, 2, \ldots.$$

It follows that

$$m^* \bigcap_{i=1}^{\infty} A_i = 0.$$

(Why?) But

$$\bigcap_{i=1}^{\infty} A_i = J(\overline{D}s = \infty).$$

(Why?) This implies that

$$m^* J(\overline{D}s = \infty) = 0,$$

and so $\overline{D}s < \infty$ on J, *except for a null set.*

But by Lemma 2 in §2, all of E^n is a countable union of such sets J (open cubes). Thus $\overline{D}s < \infty$ on $E^n - Z$, where Z is a countable union of null sets: $mZ = 0$.

As $0 \le \underline{D}s \le \overline{D}s$ on all of E^n, we have

$$0 \le \underline{D}s \le \overline{D}s < \infty \quad \text{ a.e. on } E^n,$$

as claimed. □

Lemma 2. *With the same notation, $\underline{D}s = \overline{D}s$ a.e. on E^n.*

Proof Outline. With J as in the previous proof, let

$$H = J(\overline{D}s > \underline{D}s).$$

Then H is a countable union of sets

$$H_{uv} = J(\overline{D}s > v > u > \underline{D}s)$$

over *rational* u, v. Thus it suffices to show that all such H_{uv} are m-null.

Let Q be one of them; so $Q \subseteq J$ and

$$m^* Q \le mJ < \infty.$$

Hence given $\varepsilon > 0$, there is an *open* set $G \subseteq J$ with $G \supseteq Q$ and

$$mG < m^* Q + \varepsilon.$$

(Why?) We fix this G and set

$$\mathcal{K} = \left\{ I \in \overline{\mathcal{K}} \;\middle|\; I \subseteq G,\ \frac{sI}{mI} < u \right\}.$$

By the definition of $\underline{D}s$, $\mathcal{K}$ is a V-covering of Q (verify!); so by Problem 3,

$$m^*\left(Q \cap \bigcup I_k^o\right) = m^* Q$$

for a disjoint sequence

$$\{I_k\} \subseteq \mathcal{K}, \quad \bigcup I_k \subseteq G.$$

Let

$$G' = \bigcup_{k=1}^{\infty} I_k^o$$

(an open set), and $Q_o = Q \cap G'$; so

$$m^*Q = m^*Q_o \leq mG^* \leq mG < m^*Q + \varepsilon.$$

(Explain!)

Next, let

$$\mathcal{K}' = \left\{ I \in \overline{\mathcal{K}} \;\middle|\; I \subseteq G',\ \frac{sI}{mI} > v \right\}$$

It is a V-covering of Q_o (why?); so

$$m^*\left(Q_o - \bigcup I_k'\right) = 0$$

for a disjoint sequence $\{I_k'\} \subseteq \mathcal{K}'$. Verify that

$$\begin{aligned} u \cdot (m^*Q + \varepsilon) &> u \cdot mG' = u \cdot \sum mI_k^o \\ &\geq \sum sI_k^o = sG' \\ &\geq \sum sI_k' \\ &\geq v \cdot \sum mI_k' = v \cdot m \bigcup I_k' \\ &\geq v \cdot m^*\left(Q_o \cap \bigcup I_k'\right) = v \cdot m^*Q_o = v \cdot m^*Q. \end{aligned}$$

Thus

$$(\forall\, \varepsilon > 0) \quad u \cdot (m^*Q + \varepsilon) \geq v \cdot m^*Q.$$

Making $\varepsilon \to 0$, we get

$$u \cdot m^*Q \geq v \cdot m^*Q.$$

As $u < v$, m^*A *must be* 0. This is the desired result. $\square$

Proof of Theorem 3. To fix ideas, let $f\uparrow$.

Let $s = m_f$ be the f-*induced* LS measure in E^1 (§9) so that

$$s[p, x] = f(x+) - f(p-).$$

By Lemmas 1 and 2, it suffices to show that f is differentiable at every $p \in E^1$, with

$$\underline{D}s(p) = \overline{D}s(p) \neq \infty.$$

Fix any such p and set

$$q = \underline{D}s(p) = \overline{D}s(p) \neq \infty.$$

Then f *is continuous at* p; for otherwise,

$$f(p+) - f(p-) > 0,$$

whence

$$\overline{D}s(p) = \infty.$$

(Why?) Also, by Definition 3, given $\varepsilon > 0$, there is a natural r such that

$$q - \varepsilon < g_r(p) \le h_r(p) < q + \varepsilon.$$

Let

$$x \in G_{\neg p}\left(\frac{1}{r}\right).$$

If $x > p$, then

$$\Delta x = x - p = m[p, x],$$

and by continuity,

$$\begin{aligned} \Delta f = f(x) - f(p) &\le f(x+) - f(p) \\ &= f(x+) - f(p-) = s[p, x] \\ &\le \Delta x \cdot h_r(p) < \Delta x(q + \varepsilon). \end{aligned}$$

Also, if $x > y > p$, then

$$\Delta f \ge f(y+) - f(p-) = s[p, y] \ge \Delta y \cdot g_r(p) > \Delta y(q - \varepsilon),$$

where

$$\Delta y = y - p = m[p, y].$$

Making $y \nearrow x$, with x fixed, we get

$$(q - \varepsilon)\Delta x \le \Delta f < (q + \varepsilon)\Delta x.$$

Similarly in the case $x < p$.

Thus with $\varepsilon \to 0$, we obtain

$$f'(p) = \lim_{x \to p} \frac{\Delta f}{\Delta x} = q \ne \infty. \quad \square$$

Problems on Vitali Coverings

1. Prove Theorem 1 for *globes*, filling in all details.
[Hint: Use Problem 16 in §8.]

⇒2. Show that *any* (even uncountable) union of globes or nondegenerate cubes $J_i \subset E^n$ is L-measurable.
[Hint: Include in $\mathcal{K}$ each globe (cube) that lies in some J_i. Then Theorem 1 represents $\bigcup J_i$ as a *countable* union plus a null set.]

3. Supplement Theorem 1 by proving that

$$m^*\left(A - \bigcup I_k^o\right) = 0$$

and

$$m^* A = m^*\left(A \cap \bigcup I_k^o\right);$$

here I^o = interior of I.

4. Fill in all proof details in Lemmas 1 and 2. Do it also for $\overline{\mathcal{K}} = \{globes\}$.

5. Given $mZ = 0$ and $\varepsilon > 0$, prove that there are open globes

$$G_k^* \subseteq E^n,$$

with

$$Z \subset \bigcup_{k=1}^{\infty} G_k^*$$

and

$$\sum_{k=1}^{\infty} mG_k^* < \varepsilon.$$

[Hint: Use Problem 3(f) in §5 and Problem 16(iii) from §8.]

6. Do Problem 3 in §5 for

(i) $\mathcal{C}' = \{\text{open globes}\}$, and

(ii) $\mathcal{C}' = \{\textit{all} \text{ globes in } E^n\}$.

[Hints for (i): Let m' = outer measure induced by $v' \colon \mathcal{C}' \to E^1$. From Problem 3(e) in §5, show that

$$(\forall\, A \subseteq E^n) \quad m'A \geq m^*A.$$

To prove $m'A \leq m^*A$ also, fix $\varepsilon > 0$ and an open set $G \supseteq A$ with

$$m^*A + \varepsilon \geq mG \text{ (Theorem 3 of §8).}$$

Globes inside G cover A in the V-sense (why?); so

$$A \subseteq Z \cup \bigcup G_k \; (\textit{disjoint})$$

for some globes G_k and null set Z. With G_k^* as in Problem 5,

$$m'A \leq \sum (mG_k + mG_k^*) \leq mG + \varepsilon \leq m^*A + 2\varepsilon.]$$

7. Suppose $f \colon E^n \overset{\text{onto}}{\longleftrightarrow} E^n$ is an *isometry*, i.e., satisfies

$$|f(\bar{x}) - f(\bar{y})| = |\bar{x} - \bar{y}| \quad \text{for } \bar{x}, \bar{y} \in E^n.$$

Prove that

(i) $(\forall\, A \subseteq E^n)\; m^*A = m^*f[A]$, and

(ii) $A \in \mathcal{M}^*$ iff $f[A] \in \mathcal{M}^*$.

[Hints: If A is a globe of radius r, so is $f[A]$ (verify!); thus Problems 14 and 16 in §8 apply. In the general case, argue as in Theorem 4 of §8, replacing intervals by *globes* (see Problem 6). Note that f^{-1} is an isometry, too.]

7′. From Problem 7 infer that Lebesgue measure in E^n is *rotation invariant*. (A *rotation* about $\bar{p}$ is an isometry f such that $f(\bar{p}) = \bar{p}$.)

8. A V-covering $\mathcal{K}$ of $A \subseteq E^n$ is called *normal* iff

(i) $(\forall\, I \in K)\ 0 < m\overline{I} = mI^o$, and

(ii) for every $\bar{p} \in A$, there is some $c \in (0, \infty)$ and a sequence

$$I_k \to \bar{p} \quad (\{I_k\} \subseteq \mathcal{K})$$

such that

$$(\forall\, k)\ (\exists \text{ cube } J_k \supseteq I_k) \quad c \cdot m^* I_k \geq mJ_k.$$

(We then say that $\bar{p}$ and $\{I_k\}$ are *normal*; specifically, *c-normal*.)

Prove Theorems 1 and 2 for any *normal* $\mathcal{K}$.

[Hints: By Problem 21 of Chapter 3, §16, $dI = d\overline{I}$.

First, suppose $\mathcal{K}$ is *uniformly normal*, i.e., all $\bar{p} \in A$ are c-normal for *the same* c.

In the general case, let

$$A_i = \{\bar{x} \in A \mid \bar{x} \text{ is } i\text{-normal}\}, \quad i = 1, 2, \ldots;$$

so $\mathcal{K}$ is *uniform* for A_i. Verify that $A_i \nearrow A$.

Then select, step by step, as in Theorem 1, a disjoint sequence $\{I_k\} \subseteq \mathcal{K}$ and naturals $n_1 < n_2 < \cdots < n_i < \cdots$ such that

$$(\forall\, i) \quad m^*\left(A_i - \bigcup_{k=1}^{n_i} I_k\right) < \frac{1}{i}.$$

Let

$$U = \bigcup_{k=1}^{\infty} I_k.$$

Then

$$(\forall\, i) \quad m^*(A_i - U) < \frac{1}{i}$$

and

$$A_i - U \nearrow A - U.$$

(Why?) Thus by Problems 7 and 8 in §6,

$$m^*(A - U) \leq \lim_{i \to \infty} \frac{1}{i} = 0.]$$

9. A V-covering $\overline{\mathcal{K}}^*$ of E^n is called *universal* iff

(i) $(\forall\, I \in \overline{\mathcal{K}}^*)\ 0 < m\overline{I} = mI^o < \infty$, and

(ii) whenever a *subfamily* $\mathcal{K} \subseteq \overline{\mathcal{K}}^*$ covers a set $A \subseteq E^n$ in the V-sense, we have

$$m^*\Bigl(A - \bigcup I_k\Bigr) = 0$$

for a disjoint sequence

$$\{I_k\} \subseteq \mathcal{K}.$$

Show the following.

(a) $\overline{\mathcal{K}}^* \subseteq \mathcal{M}^*$.

(b) Lemmas 1 and 2 are true with $\overline{\mathcal{K}}$ replaced by any universal $\overline{\mathcal{K}}^*$. (In this case, write $\underline{D}^*s$ and $\overline{D}^*s$ for the analogues of $\underline{D}s$ and $\overline{D}s$.)

(c) $\underline{D}s = \underline{D}^*s = \overline{D}^*s = \overline{D}s$ a.e.

[Hints: (a) By (i), $I = \overline{I}$ minus a null set $Z \subseteq \overline{I} - I^o$.

(c) Argue as in Lemma 2, but set

$$Q = J(\underline{D}^*s > u > v > \underline{D}s)$$

and

$$\mathcal{K}' = \left\{ I \in \overline{\mathcal{K}}^* \;\middle|\; I \subseteq G',\ \frac{sI}{mI} > v \right\}$$

to prove a.e. that $\underline{D}^*s \le \underline{D}s$; similarly for $\underline{D}s \le D^*s$.

Throughout assume that $s\colon \mathcal{M}' \to E^*$ $(\mathcal{M}' \supseteq \overline{\mathcal{K}} \cup \overline{\mathcal{K}}^*)$ is a measure in E^n, finite on $\overline{\mathcal{K}} \cup \overline{\mathcal{K}}^*$.]

10. Continuing Problems 8 and 9, verify that

(a) $\overline{\mathcal{K}} = \{$nondegenerate cubes$\}$ is a normal and universal V-covering of E^n;

(b) so also is $\overline{\mathcal{K}}^o = \{$all globes in $E^n\}$;

(c) $\overline{\mathcal{C}} = \{$nondegenerate intervals$\}$ is normal.

Note that $\overline{\mathcal{C}}$ is *not* universal.[1]

11. Continuing Definition 3, we call q a *derivate* of s, and write $q \sim Ds(\bar{p})$, iff

$$q = \lim_{k\to\infty} \frac{sI_k}{mI_k}$$

for some sequence $I_k \to \bar{p}$, with $I_k \in \overline{\mathcal{K}}$.

Set

$$D_{\bar{p}} = \{q \in E^* \mid q \sim Ds(\bar{p})\}$$

and prove that

$$\underline{D}s(\bar{p}) = \min D_{\bar{p}} \text{ and } \overline{D}s(\bar{p}) = \max D_{\bar{p}}.$$

---

[1] See M. E. Munroe, *Measure and Integration*, Addison–Wesley (1971), pp. 173–175.

12. Let $\mathcal{K}^*$ be a *normal* V-covering of E^n (see Problem 8). Given a measure s in E^n, finite on $\mathcal{K}^* \cup \overline{\mathcal{K}}$, write

$$q \sim D^* s(\bar{p})$$

iff

$$q = \lim_{k\to\infty} \frac{sI_k}{mI_k}$$

for some *normal* sequence $I_k \to \bar{p}$, with $I_k \in \mathcal{K}^*$.

Set

$$D^*_{\bar{p}} = \{q \in E^* \mid q \sim D^* s(\bar{p})\},$$

and then

$$\underline{D}^* s(\bar{p}) = \inf D^*_{\bar{p}} \text{ and } \overline{D}^* s(\bar{p}) = \sup D^*_{\bar{p}}.$$

Prove that

$$\underline{D}s = \underline{D}^* s = \overline{D}^* s = \overline{D}s \text{ a.e. on } E^n.$$

[Hint: $E^n = \bigcup_{i=1}^{\infty} E_i$, where

$$E_i = \{\bar{x} \in E^n \mid \bar{x} \text{ is } i\text{-normal}\}.$$

On each E_i, $\mathcal{K}^*$ is *uniformly* normal. To prove $\underline{D}s = \underline{D}^* s$ a.e. on E_i, "imitate" Problem 9(c). Proceed.]

*§11. Generalized Measures. Absolute Continuity

I. We now return to *general* set functions $s\colon \mathcal{M} \to E$, with E as in Definition 1 of §4.

Definition 1.

A set function $s\colon \mathcal{M} \to E$ is a *generalized measure* in a set S, and $(S, \mathcal{M}, s)$ is a *generalized measure space*, iff s is σ-additive and semifinite (i.e., $s \neq +\infty$ or $s \neq -\infty$) on $\mathcal{M}$, a σ-ring in S, and $s\emptyset = 0$.

We call s a *signed measure* iff $E \subseteq E^*$ (i.e., s is *real* or *extended real*); if $s \geq 0$ then s is a *measure*; s may also be complex ($E = C$) or vector valued.

Definition 2.

Given a set function $s\colon \mathcal{M} \to E$, we define its *total variation*

$$v_s\colon \mathcal{M} \to [0, \infty]$$

by

$$(\forall A \in \mathcal{M}) \quad v_s A = \sup \sum_i |sX_i|,$$

taking the sup over all countable *disjoint* subfamilies $\{X_i\} \subseteq \mathcal{M}$ with $\bigcup_i X_i \subseteq A$.

Note 1. If $\mathcal{M}$ is a *σ-ring*, we may equivalently require that

$$\bigcup X_i = A$$

with $\{X_i\}$ a disjoint sequence in $\mathcal{M}$ (add the term $X_o = A - \bigcup_i X_i$ if necessary).[1]

Corollary 1. *If s and v_s are as in Definition 2, then*

(i) *v_s is monotone on $\mathcal{M}$, and*

(ii) *$|sA| \leq v_s A$ for every $A \in \mathcal{M}$.*

Proof. For (i), let $A \subseteq B$, $A, B \in \mathcal{M}$. Take any disjoint sequence $\{X_i\} \subseteq \mathcal{M}$, with

$$\bigcup X_i \subseteq A \subseteq B.$$

By definition,

$$\sum_i |sX_i| \leq v_s B.$$

Thus $v_s B$ is an upper bound of all such sums, with $\bigcup X_i \subseteq A$. Hence

$$v_s A = \operatorname{lub} \sum |sX_i| \leq v_s B,$$

proving (i).

To prove (ii), just let $\{X_i\}$ consist of A alone. $\square$

Theorem 1. *If $s\colon \mathcal{M} \to E$ is a generalized measure, then v_s is a* measure *on $\mathcal{M}$.*

Proof. By definition, $v_s \geq 0$ on $\mathcal{M}$, a σ-ring, and $v_s \emptyset = 0$. (Why?) It remains to prove σ-additivity.

Thus let

$$A = \bigcup_n A_n \ (\textit{disjoint}),$$

with $A, A_n \in \mathcal{M}$. To show that

$$v_s A = \sum_n v_s A_n,$$

take any $\mathcal{M}$-partition $\{X_i\}$ of A. Then

$$(\forall\, i) \quad X_i = X_i \cap A = X_i \cap \bigcup_n A_n = \bigcup_n (X_i \cap A_n) \ (\textit{disjoint}).$$

---

[1] Any such $\{X_i\}$ is called an *$\mathcal{M}$-partition* of A (Chapter 8, §1); it may consist of A *alone*.

Similarly,

$$(\forall\, n) \quad A_n = \bigcup_i (A_n \cap X_i);$$

so by definition,

$$(\forall\, n) \quad \sum_i |s(A_n \cap X_i)| \le v_s A_n.$$

Hence as

$$X_i = \bigcup_n (X_i \cap A_n),$$

we get

$$\begin{aligned}\sum_i |sX_i| = \sum_i \left| s \bigcup_n (A_n \cap X_i) \right| &= \sum_i \left| \sum_n s(A_n \cap X_i) \right| \\ &\le \sum_{n,i} |s(A_n \cap X_i)| \le \sum_n v_s A_n.\end{aligned}$$

As $\{X_i\}$ was an *arbitrary* $\mathcal{M}$-partition of A,

$$v_s A = \sup \sum |sX_i| \le \sum_n v_s A_n.$$

It remains to show that

$$\sum_n v_s A_n \le v_s A.$$

This is trivial if $v_s A = \infty$.

Thus let $v_s A < \infty$. Then

$$(\forall\, n) \quad v_s A_n \le v_s A < \infty$$

by Corollary 1(i). Now fix $\varepsilon > 0$. By properties of lub, each A_n has an $\mathcal{M}$-partition,

$$A_n = \bigcup_k X_{nk},$$

such that

$$v_s A_n - \frac{\varepsilon}{2^n} < \sum_k |sX_{nk}|.$$

All X_{nk} *combined* (for all n and k) form an $\mathcal{M}$-partition of A. Thus by definition,

$$v_s A \ge \sum_n \sum_k |sX_{nk}| \ge \sum_n \Big(v_s A_n - \frac{\varepsilon}{2^n} \Big) \ge \sum_n v_s A_n - \varepsilon.$$

With $\varepsilon \to 0$, we get

$$\sum_n v_s A_n \le v_s A,$$

as required. □

Definition 3.

Given

$$s\colon \mathcal{M} \to E \text{ and } t\colon \mathcal{M}' \to E',^2$$

we say that s is

(i) *t-continuous* (written $s \ll t$) iff

$$v_t X = 0 \implies |sX| = 0 \quad (X \in \mathcal{M}');$$

(ii) *absolutely t-continuous* (or absolutely continuous with respect to t) iff

$$v_t X \to 0 \implies sX \to 0,$$

i.e.,

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, X \in \mathcal{M}') \quad v_t X < \delta \implies |sX| < \varepsilon;$$

(iii) *t-finite* iff

$$v_t X < \infty \implies |sX| < \infty \quad (X \in \mathcal{M}').$$

Corollary 2. *If two set functions $s, u\colon \mathcal{M} \to E$ are t-continuous (absolutely t-continuous) so are $s \pm u$, and so is ks for any k from the scalar field of E.*[3]

The proof is left to the reader. (Use Definition 3(i)(ii), *quantified* formula.)

Theorem 2. *Let $s\colon \mathcal{M} \to E$ and $t\colon \mathcal{M}' \to E'$.*

(i) *If $s \ll t$, then $v_s \ll t$.*

(ii) *If, in addition, s and t are generalized measures and v_s is t-finite, then both v_s and s are absolutely t-continuous.*

(iii) *$v_s \ll t$ implies $s \ll t$ (which is obvious).*

Proof. Fix $A \in \mathcal{M}$ and any disjoint sequence $X_i \in \mathcal{M}$, with

$$\bigcup X_i \subseteq A.$$

If $v_t A = 0$, then (Corollary 1)

$$(\forall\, i) \quad v_t X_i = 0;$$

---

[2] For the rest of this section, we assume that $\mathcal{M}$ and $\mathcal{M}'$ satisfy $X \in \mathcal{M}$ whenever $X \in \mathcal{M}'$ and $v_t X < \infty$.

[3] If $E = E^*$, we assume $k \in E^1$. If s is scalar valued, k may be a *vector* in E.

so by the t-continuity of s, $|sX_i| = 0$, and hence $\sum |sX_i| = 0$. As this holds for *any* such sum, we also have

$$v_s A = \sup \sum |sX_i| = 0$$

whenever $v_t A = 0$. This proves assertion (i).

Now, let s and t be as in (ii); so v_s and v_t are *measures* by Theorem 1. Suppose v_s is *not* absolutely t-continuous. Then

$$(\exists\, \varepsilon > 0)\ (\forall\, \delta > 0)\ (\exists\, X \in \mathcal{M}') \quad v_t X < \delta \text{ and } v_s X \geq \varepsilon.$$

(Why?) Taking

$$\delta_n = 2^{-n},$$

fix $(\forall\, n)$ a set $X_n \in \mathcal{M}'$, with

$$v_t X_n < 2^{-n} \text{ and } v_s X_n \geq \varepsilon.$$

Let

$$Y_n = \bigcup_{k=n}^{\infty} X_k \text{ and } Y = \bigcap_{n=1}^{\infty} Y_n;$$

so $Y, Y_n \in \mathcal{M}'$, $Y_n \searrow Y$, and

$$v_t Y_n \leq \sum_{k=n}^{\infty} v_t X_k < \sum_{k=n}^{\infty} 2^{-k} \leq 2^{1-n}.$$

Thus by Theorem 2 in §4 (right continuity),

$$v_t Y = \lim_{n\to\infty} v_t Y_n \leq \lim_{n\to\infty} 2^{1-n} = 0.$$

Hence by the t-continuity of v_s (see (i)),

$$v_s Y = 0 < \varepsilon.$$

On the other hand, as $Y_n \supseteq X_n$, we have

$$v_s Y_n \geq v_s X_n \geq \varepsilon.$$

Also, $v_t Y_n \leq 2^{1-n}$ implies $v_s Y_n < \infty$ (v_s is t-finite). Hence

$$v_s Y = \lim_{n\to\infty} v_s Y_n \geq \varepsilon,$$

a contradiction. Thus v_s *is* absolutely t-continuous.

So is s; for by Corollary 1(ii), we have

$$(\forall\, \varepsilon > 0)\ (\exists\, \delta > 0)\ (\forall\, X \in \mathcal{M}') \quad v_t X < \delta \Longrightarrow |sX| \leq v_s X < \varepsilon,$$

proving (ii). $\square$

Note 2. Absolute t-continuity *always* implies t-continuity.[4]

II. Special notions apply to *signed* measures. First of all, we have the following definition.

Definition 4.

A set $A \subseteq S$ in a signed measure space $(S, \mathcal{M}, s)$ is called *positive* (*negative*) iff $sX \geq 0$ ($sX \leq 0$, respectively) whenever

$$A \supseteq X, \ X \in \mathcal{M}.$$

We set

$$\mathcal{M}^+ = \{X \in \mathcal{M} \mid X \text{ is positive}\}$$

and

$$\mathcal{M}^- = \{X \in \mathcal{M} \mid X \text{ is negative}\}.$$

The easy proof of Lemmas 1 and 2 is left to the reader.

Lemma 1. *In any signed measure space, $\mathcal{M}^+$ and $\mathcal{M}^-$ are σ-rings.*

Lemma 2. *If s, t are signed measures on $\mathcal{M}$, then*

(i) *so is ks ($k \in E^1$);*

(ii) *so also are $s \pm t$, provided s or t is finite on $\mathcal{M}$.*

Note 3. Lemma 2 applies to generalized measures $s, t \colon \mathcal{M} \to E$ as well.

Lemma 3. *Let $s \colon \mathcal{M} \to E^*$ be a signed measure. Let $A \in \mathcal{M}$, $0 < sA < \infty$. Then A has a subset $Q \in \mathcal{M}^+$ such that*

$$0 < sA \leq sQ < \infty.$$

Proof. If $A \in \mathcal{M}^+$, take $Q = A$.

Otherwise, A has subsets of *negative* measure. Let then n_1 be the *least* natural for which there is a set $A_1 \in \mathcal{M}$, with

$$A_1 \subseteq A \text{ and } sA_1 < -\frac{1}{n_1}.$$

(why does such n_1 exist?); then

$$s(A - A_1) > sA > 0.$$

Now, if $A - A_1 \in \mathcal{M}^+$, take $Q = A - A_1$. If not, let n_2 be the *least* natural for which there is $A_2 \in \mathcal{M}$, with

$$A_2 \subseteq A - A_1 \text{ and } sA_2 < -\frac{1}{n_2}.$$

---

[4] For if $v_t X = 0$, then $v_t X < \delta$ for *any* $\delta > 0$. Thus Definition 3(ii) implies $(\forall \varepsilon > 0)$ $|sX| < \varepsilon$; hence $|sX| = 0$.

Again, if

$$A - \bigcup_{i=1}^{2} A_i$$

is positive, put

$$Q = A - \bigcup_{i=1}^{2} A_i.$$

If not, let n_3 be the *least* natural for which there is $A_3 \in \mathcal{M}$, with

$$A_3 \subseteq A - \bigcup_{i=1}^{2} A_i$$

and

$$sA_3 < -\frac{1}{n_3}.$$

Continuing, we either find the desired Q at some step or obtain a sequence $\{A_k\} \subseteq \mathcal{M}$ such that

$$(1) \qquad (\forall\, k \in N) \quad sA_k < -\frac{1}{n_k} \text{ and } A_{k+1} \subseteq A - \bigcup_{i=1}^{k} A_i$$

(so the A_k are *disjoint*). In the latter case, let

$$Q = A - \bigcup_{k=1}^{\infty} A_k;$$

so

$$A = Q \cup \bigcup_{k=1}^{\infty} A_k \ (\textit{disjoint}),$$

and

$$sQ + \sum_k sA_k = sA.$$

As $|sA| < \infty$ (by assumption), $\sum sA_k$ *converges*. By (1), then,

$$\sum_k \frac{1}{n_k} \leq \sum_k (-sA_k) < \infty.$$

Therefore,

$$\lim_{k\to\infty} \frac{1}{n_k} = 0,$$

i.e.,

$$\lim_{k\to\infty} n_k = \infty.$$

Also, as $sA_k < 0$ and $sA > 0$, we have

$$sQ = sA - \sum sA_k > sA > 0.$$

Now, given $\varepsilon > 0$, choose k so large that

$$\varepsilon > \frac{1}{n_k - 1}.$$

As

$$Q \subseteq A - \bigcup_{i=1}^{k} A_i,$$

our definition of the n_k implies that Q can have no subsets $X \in \mathcal{M}$, with

$$sX < -\varepsilon < -\frac{1}{n_k - 1}.$$

(Why?) As ε is arbitrary, Q has *no* subsets of negative measure.

Thus $Q \in \mathcal{M}^+$, $Q \subseteq A$, and

$$0 < sA \leq sQ < \infty,$$

as required. □

The following theorem is named after the mathematician Hans Hahn.

Theorem 3 (Hahn decomposition theorem). *In any signed measure space $(S, \mathcal{M}, s)$, there is a positive set $P \subseteq S$ whose complement is negative. Moreover, P or $-P$ can be chosen* from $\mathcal{M}$, *according to whether $s \neq \infty$ or $s \neq -\infty$ on $\mathcal{M}$.*

If $S \in \mathcal{M}$, both *P and $-P$ can be made s-measurable:*

$$P \in \mathcal{M}^+ \text{ and } -P \in \mathcal{M}^-.$$

Proof. By definition, s is semifinite; so $s \neq \infty$ or $s \neq -\infty$ on $\mathcal{M}$; say, $s \neq +\infty$.

As $\mathcal{M}^+$ is a σ-ring (Lemma 1), the restriction of s to $\mathcal{M}^+$ is a *measure*, with

$$0 \leq s < \infty$$

on $\mathcal{M}^+$. Thus by Problem 13 in §6, we fix a set $P \in \mathcal{M}^+$ such that

$$sP = \max\{sX \mid X \in \mathcal{M}^+\} < \infty.$$

By Lemma 3, $sP = \max sX$, even on *all* of $\mathcal{M}$.

It remains to show that $-P$ is negative. Suppose it is not. Then $-P$ has a subset $Y \in \mathcal{M}$, with $sY > 0$; so

$$Y \cap P = \emptyset \text{ and } Y \cup P \in \mathcal{M}.$$

By additivity,

$$s(Y \cup P) = sY + sP > sP,$$

contrary to the maximality of sP. This contradiction settles the case $s \neq +\infty$.

In case $s \neq -\infty$, consider $-s$, which by Lemma 2 is likewise a signed measure, with $-s \neq +\infty$. By what was proved above, there is a set $P' \in \mathcal{M}$ that is positive *for* $-s$ (hence negative *for* s), and whose complement is positive *for* s.

Finally, if $S \in \mathcal{M}$, then $P \in \mathcal{M}$ implies

$$S - P = -P \in \mathcal{M};$$

so *both* P and $-P$ are in $\mathcal{M}$. Thus all is proved. $\square$

Note 4. The set P in Theorem 3 is not unique. However, if $P' \in \mathcal{M}^+$ is another such set, then

$$s(P - P') = 0 = s(P' - P),$$

i.e., any two such sets *can differ by a set of measure* 0 *only*. Indeed,

$$P - P' \subseteq P \text{ and } P - P' \subseteq -P';$$

so $s(P - P')$ is both ≥ 0 and ≤ 0. Thus $s(P - P') = 0$. Similarly for $P' - P$.

Theorem 4 (Jordan decomposition). *Every signed measure* $s\colon \mathcal{M} \to E^*$ *is the difference of two* measures*,*

$$s = s^+ - s^- \quad (s^+, s^- \geq 0),$$

with s^+ *or* s^- bounded *on* $\mathcal{M}$.

Proof. Suppose $s \neq +\infty$ on $\mathcal{M}$. Then by Theorem 3, there is a set $P \in \mathcal{M}^+$ such that $-P$ is negative and $sP < \infty$. Now define, for all sets $A \in \mathcal{M}$,

$$s^+A = s(A \cap P) \quad \text{and } s^-A = -s(A - P). \tag{2}$$

By additivity,

$$sA = s(A \cap P) + s(A - P) = s^+A - s^-A;$$

so $s = s^+ - s^-$ on $\mathcal{M}$, as required. Moreover,

$$s^+A = s(A \cap P) \geq 0,$$

since $A \cap P \subseteq P$ and P is positive. Similarly,

$$s^-A = -s(A - P) \geq 0,$$

since $A - P \subseteq -P$ and $-P$ is negative. Thus $s^+, s^- \geq 0$ on $\mathcal{M}$, a σ-ring.

The σ-additivity of s^+ and s^- easily follows from that of s (we leave the proof to the reader). Thus s^+ and s^- are *measures*.

Finally, by (2),

$$s^+A = s(A \cap P) \leq sP < \infty$$

for all $A \in \mathcal{M}$ (for

$$sP = \max\{sX \mid X \in \mathcal{M}\};$$

see the proof of Theorem 3). Thus s^+ is bounded, and all is proved.

The case $s \neq -\infty$ is similar. □

Note 5. For any set $X \subseteq A$ $(X \in \mathcal{M})$, we have

$$sX = s^+X - s^-X \leq s^+X \leq s^+A,$$

for s^+ and s^- are ≥ 0 and *monotone*. Thus s^+A is an upper bound of

$$\{sX \mid A \supseteq X \in \mathcal{M}\}.$$

By (2), this bound is *reached* when $X = A \cap P$; so it is a *maximum*. Similarly for s^-; thus

$$(3) \quad s^+A = \max\{sX \mid A \supseteq X \in \mathcal{M}\} \text{ and } s^-A = \max\{-sX \mid A \supseteq X \in \mathcal{M}\}.$$

Note 6. The decomposition is not unique, for we also have

$$s = (s^+ + m) - (s^- + m)$$

for any finite measure m on $\mathcal{M}$. However, it becomes unique if we add condition (3). When so defined, s^+ and s^- are called the *Jordan components* of s.

Note 7. Formula (2) shows that

$$(-s)^+ = s^- \text{ and } (-s)^- = s^+.$$

Corollary 3. *With s, s^+, and s^- as in* (3), *we have the following.*

(i) $v_s = s^+ + s^-$; *hence if s is a* measure $(s^- = 0)$, *then*

$$s = v_s = s^+.$$

(ii) v_s *is finite* (*t-finite*, *t-continuous*, *absolutely t-continuous*) *iff s^+ and s^- are*, *i.e.*, *iff s is.*

Proof. We give only an outline here.

(i) Take any $\mathcal{M}$-partition

$$A = \bigcup X_i \ (disjoint).$$

Setting

$$m = s^+ + s^-,$$

verify that

$$|sX_i| \leq mX_i$$

and

$$\sum |sX_i| \leq \sum mX_i = m\bigcup X_i = mA.$$

Thus mA is an upper bound of sums

$$\sum |sX_i|.$$

This bound is *reached* when $X_1 = A \cap P$, $X_2 = A - P$ (P as in (2)).

(ii) Use Theorem 2, Corollary 2, and Definition 3. Note that $v_s \geq |s|$, s^+, and s^-. □

Corollary 4. *A t-finite signed measure s is absolutely t-continuous iff it is t-continuous.*

In particular, this applies to *finite* measures.

Corollary 4 follows from Theorem 2 and Note 2, by Corollary 3.

III. If $E = E^n$ (C^n), the function

$$s\colon \mathcal{M} \to E$$

has n real (complex) components

$$s_1, \ldots, s_n,$$

as defined in Chapter 4, §3. As in Theorem 2 of Chapter 4, §3, one easily obtains the following.

Theorem 5. *A set function $s\colon \mathcal{M} \to E^n$ (C^n) is t-continuous (absolutely t-continuous, additive, σ-additive) iff its n components are. Hence a complex set function s is t-continuous (etc.) iff its real and imaginary parts are.*

For σ-additivity, one can argue as follows. Let

$$A = \bigcup_{i=1}^{\infty} A_i \ (\textit{disjoint}),$$

with $A, A_i \in \mathcal{M}$. Use Theorem 2 in Chapter 3, §15, with $\bar{p} = sA$ and

$$\bar{x}_m = \sum_{i=1}^{m} sA_i,$$

to get $p_k = s_k A$, and

$$x_{mk} = \sum_{i=1}^{m} s_k A_i, \quad k = 1, \ldots, n.$$

Theorem 6. *A generalized measure $s\colon \mathcal{M} \to E^n$ (C^n) is t-continuous iff it is absolutely t-continuous. It is* always *bounded on $\mathcal{M}$, as is v_s.*

Proof. As $s\colon \mathcal{M} \to E^n$ is σ-additive, so is each of its components s_k, by Theorem 5. Thus each s_k is a *finite* (real) signed measure, with

$$s_k = s_k^+ - s_k^-,$$

as in Theorem 4. Here the measures s_k^+ and s_k^- are both finite (as s is).

Thus by Problem 13 in §6, they are *bounded*, say, $s_k^+ \leq K_1$ and $s_k^- \leq K_2$ on $\mathcal{M}$. Hence by Corollaries 1 and 3,

$$|s_k| \leq v_{s_k} = s_k^+ + s_K^- \leq K_1 + K_2;$$

that is, v_{s_k} is bounded on $\mathcal{M}$ $(k = 1, 2, \dots, n)$. Hence so are s and v_s, for

$$|s| \leq v_s \leq \sum_k v_{s_k}$$

(see Problem 4(iii)).

Now, as v_s is finite, it is certainly t-finite. Thus by Theorem 2 and Note 2, s is t-continuous iff it is absolutely t-continuous.

This settles the case $E = E^n$, hence also $E = C = E^2$. The case $E = C^n$ is analogous. ☐

IV. Completion of a Generalized Measure. From Problems 14 and 15 of §6, recall that every *measure m* has a completion $\overline{m}$. A similar construction, which we now describe, applies to *generalized measures* $s\colon \mathcal{M} \to E$.

Given such an s, let $\overline{\mathcal{M}}$ be the family of all sets $X \cup Z$, where $X \in \mathcal{M}$ and Z is v_s*-null*, i.e., $Z \subseteq U$ for some $U \in \mathcal{M}$, $v_s U = 0$ (note that v_s is a *measure* here, by Theorem 2). That is,

$$\overline{\mathcal{M}} = \{X \cup Z \mid X \in \mathcal{M},\ Z \subseteq U,\ U \in \mathcal{M},\ v_s U = 0\}.$$

We now define $\bar{s}\colon \overline{\mathcal{M}} \to E$ by setting

$$\bar{s}A = sX$$

whenever $A = X \cup Z$, with X and Z as above.

As in Problems 14 and 15 of §6, it follows that $\overline{\mathcal{M}}$ is a σ-ring $\supseteq \mathcal{M}$, and that $\bar{s}$ is a σ-additive extension of s, hence a generalized measure. We call $\bar{s}$ the *completion* of s. It is *complete* in the sense that $\overline{\mathcal{M}}$ contains all v_s*-null* sets (but it may miss some subsets of X with $sX = 0$). If $s \geq 0$ (a *measure*), then $s = v_s$; so our present definitions agree with Problem 15 in §6. We use these ideas in the following part.

V. Signed Lebesgue–Stieltjes (LS) Measures. Motived by Theorem 3 in Chapter 5, §7, we shall say that a function

$$\alpha\colon E^1 \to E^1$$

is of *bounded variation on* E^1 iff

$$\alpha = g - h,$$

with $g\uparrow$ and $h\uparrow$ on all of E^1.

Then g and h induce two LS measures m_g and m_h in E^1.

Let μ_g and μ_h be their restrictions to the Borel field $\mathcal{B}$ in E^1. Then

$$\sigma_\alpha^* = \mu_g - \mu_h$$

is finite for sets $X \in \mathcal{B}$ inside any finite interval $I \subset E^1$ (as μ_g and μ_h are finite on intervals).

By Lemma 2, σ_α^* is a signed measure on the $\mathcal{B}$-sets in I. Moreover, σ_α^* *does not depend on the particular choice of* $g\uparrow$ *and* $h\uparrow$ $(g - h = \alpha)$ on I. For if also $\alpha = u - v$ $(u\uparrow,\ v\uparrow)$ on E^1, set

$$\sigma_\alpha' = \mu_u - \mu_v.$$

Then for any $(x, y] \subseteq I$,

$$\sigma_\alpha'(x, y] = \alpha(y+) - \alpha(x+) = \sigma_\alpha^*(x, y] \quad \text{(verify!)};$$

so by Problem 13 in §5, $\sigma_\alpha' = \sigma_\alpha^*$ on $\mathcal{B}$-*sets* in I.

Thus σ_α^* is uniquely determined by α. Its completion

$$s_\alpha = \overline{\sigma_\alpha^*}$$

is the α-*induced Lebesgue–Stieltjes* (*LS*) *signed measure in* I.

If further μ_g or μ_h is finite on all of $\mathcal{B}$, the same process defines a signed LS measure in all of E^1.

Problems on Generalized Measures

1. Complete the proofs of Theorems 1, 4, and 5.

1′. Do it also for the lemmas and Corollary 3.

2. Verify the following.

(i) In Definition 2, one can equivalently replace "countable $\{X_i\}$" by "finite $\{X_i\}$."

(ii) If $\mathcal{M}$ is a *ring*, Note 1 holds for *finite* sequences $\{X_i\}$.

(iii) If $s\colon \mathcal{M} \to E$ is additive on $\mathcal{M}$, a *semiring*, so is v_s.
[Hint: Use Theorem 1 from §4.]

3. For any set functions s, t on $\mathcal{M}$, prove that

(i) $v_{|s|} = v_s$, and

(ii) $v_{st} \leq a v_t$, provided st is *defined* and

$$a = \sup\{|sX| \mid X \in \mathcal{M}\}.$$

4. Given $s, t \colon \mathcal{M} \to E$, show that

(i) $v_{s+t} \leq v_s + v_t$;

(ii) $v_{ks} = |k| v_s$ (k as in Corollary 2); and

(iii) if $E = E^n$ (C^n) and

$$s = \sum_{k=1}^{n} s_k \bar{e}_k,$$

then

$$v_{s_k} \leq v_s \leq \sum_{k=1}^{n} v_{sk}.$$

[Hints: (i) If

$$A \supseteq \bigcup X_i \ (\textit{disjoint}),$$

with $A_i, X_i \in \mathcal{M}$, verify that

$$|(s+t)X_i| \leq |sX_i| + |tX_i|,$$

$$\sum |(s+t)X_i| \leq v_s A + v_t A, \text{ etc.};$$

(ii) is analogous.

(iii) Use (ii) and (i), with $|\bar{e}_k| = 1$.]

5. If $g\uparrow$, $h\uparrow$, and $\alpha = g - h$ on E^1, can one define the signed LS measure s_α by simply setting $s_\alpha = m_g - m_h$ (assuming $m_h < \infty$)?
[Hint: the domains of m_g and m_h may be *different*. Give an example. How about taking their intersection?]

6. Find an LS measure m_α such that α is continuous and one-to-one, but m_α is not m-finite (m = Lebesgue measure).
[Hint: Take

$$\alpha(x) = \begin{cases} \dfrac{x^3}{|x|}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and

$$A = \bigcup_{n=1}^{\infty} \left(n, n + \frac{1}{n^2}\right].]$$

7. Construct complex and vector-valued LS measures $s_\alpha \colon \mathcal{M}_\alpha^* \to E^n$ (C^n) in E^1.

8. Show that if $s\colon \mathcal{M} \to E^n$ (C^n) is additive and bounded on $\mathcal{M}$, a *ring*, so is v_s.

[Hint: By Problem 4(iii), reduce all to the *real* case.

Use Problem 2. Given a finite disjoint sequence $\{X_i\} \subseteq \mathcal{M}$, let U^+ (U^-) be the union of those X_i for which $sX_i \geq 0$ ($sX_i < 0$, respectively). Show that

$$\sum sX_i = sU^+ - sU^- \leq 2\sup|s| < \infty.]$$

9. For *any* $s\colon \mathcal{M} \to E^*$ and $A \in \mathcal{M}$, set

$$s^+A = \sup\{sX \mid A \supseteq X \in \mathcal{M}\}$$

and

$$s^-A = \sup\{-sX \mid A \supseteq X \in \mathcal{M}\}.$$

Prove that if s is additive and bounded on $\mathcal{M}$, a *ring*, so are s^+ and s^-; furthermore,

$$\begin{aligned} s^+ &= \frac{1}{2}(v_s + s) \geq 0, \\ s^- &= \frac{1}{2}(v_s - s) \geq 0, \\ s &= s^+ - s^-, \text{ and} \\ v_s &= s^+ + s^-. \end{aligned}$$

[Hints: Use Problem 8. Set

$$s' = \frac{1}{2}(v_s + s).$$

Then $(\forall X \in \mathcal{M} \mid X \subseteq A)$

$$\begin{aligned} 2sX = sA + sX - s(A - X) &\leq sA + (|sX| + |s(A - X)|) \\ &\leq sA + v_sA = 2s'A. \end{aligned}$$

Deduce that $s^+A \leq s'A$.

To prove also that $s'A \leq s^+A$, let $\varepsilon > 0$. By Problems 2 and 8, fix $\{X_i\} \subseteq \mathcal{M}$, with

$$A = \bigcup_{i=1}^{n} X_i \ (\textit{disjoint})$$

and

$$v_sA - \varepsilon < \sum_{i=1}^{n} |sX_i|.$$

Show that

$$2s'A - \varepsilon = v_sA + sA - \varepsilon \leq sU^+ - sU^- + s\bigcup_{i=1}^{n} X_i = 2sU^+$$

and

$$2s^+A \geq 2sU^+ \geq 2s'A - \varepsilon.]$$

10. Let

$$\mathcal{K} = \{\textit{compact} \text{ sets in a topological space } (S, \mathcal{G})\}$$

(adopt Theorem 2 in Chapter 4, §7, as a *definition*). Given

$$s\colon \mathcal{M} \to E, \quad \mathcal{M} \subseteq 2^S,$$

we call s *compact regular* (CR) iff

$$\begin{aligned}&(\forall\, \varepsilon > 0)\ (\forall\, A \in \mathcal{M})\ (\exists\, F \in \mathcal{K})\ (\exists\, G \in \mathcal{G})\\ &\qquad F, G \in \mathcal{M},\ F \subseteq A \subseteq G, \text{ and } v_s G - \varepsilon \le v_s A \le v_s F + \varepsilon.\end{aligned}$$

Prove the following.

(i) If $s, t\colon \mathcal{M} \to E$ are CR, so are $s \pm t$ and ks (k as in Corollary 2).

(ii) If s is additive and CR on $\mathcal{M}$, a semiring, so is its extension to the ring $\mathcal{M}_s$ (Theorem 1 in §4 and Theorem 4 of §3).

(iii) If $E = E^n\,(C^n)$ and $v_s < \infty$ on $\mathcal{M}$, a *ring*, then s is CR iff its components s_k are, or in the case $E = E^1$, iff s^+ and s^- are (see Problem 9).

[Hint for (iii): Use (i) and Problem 4(iii). Consider $v_s(G - F)$.]

11. (Aleksandrov.) Show that if $s\colon \mathcal{M} \to E$ is CR (see Problem 10) and additive on $\mathcal{M}$, a ring in a topological space S, and if $v_s < \infty$ on $\mathcal{M}$, then v_s and s are σ-additive, and v_s has a unique σ-additive extension $\bar{v}_s$ to the σ-ring $\mathcal{N}$ generated by $\mathcal{M}$.

The latter holds for s, too, if $S \in \mathcal{M}$ and $E = E^n\,(C^n)$.

[Proof outline: The σ-additivity of v_s results as in Theorem 1 of §2 (first check Lemma 1 in §1 for v_s).

For the σ-additivity *of* s, let

$$A = \bigcup_{i=1}^{\infty} A_i \ (\textit{disjoint}), \quad A, A_i \in \mathcal{M};$$

then

$$\left|sA - \sum_{i=1}^{r-1} sA_i\right| \le \sum_{i=r}^{\infty} v_s A_i \to 0$$

as $r \to \infty$, for

$$\sum_{i=1}^{\infty} v_s A_i = v_s \bigcup_{i=1}^{\infty} A_i < \infty.$$

(Explain!) Now, Theorem 2 of §6 extends v_s to a *measure* on a σ-field

$$\mathcal{M}^* \supseteq \mathcal{N} \supseteq \mathcal{M}$$

(use the minimality of $\mathcal{N}$). Its restriction to $\mathcal{N}$ is the desired $\bar{v}_s$ (*unique* by Problem 15 in §6).

A similar proof holds for s, too, if $s\colon \mathcal{M} \to [0, \infty)$. The case $s : \mathcal{M} \to E^n\,(C^n)$ results via Theorem 5 and Problem 10(iii) *provided* $S \in \mathcal{M}$; for then by Corollary 1, $v_s S < \infty$ ensures the finiteness of v_s, s^+, and s^- *even on* $\mathcal{N}$.]

12. Do Problem 11 for *semirings* $\mathcal{M}$.
[Hint: Use Problem 10(ii).]

*§12. Differentiation of Set Functions

In the proof of Theorem 3 in §10 and the lemmas of that section, we saw the connection between quotients of the form

$$\frac{\Delta f}{\Delta x} = \frac{f(x) - f(p)}{x - p}$$

and those of the form

$$\frac{sI}{mI},$$

where m is Lebesgue measure and s is another suitable measure. With this in mind, we now use quotients sI/mI for forming derivatives of *set functions*.

Below, m is Lebesgue measure in E^n;

$$\overline{\mathcal{K}} = \{\text{nondegenerate cubes}\}.$$

Definition 1.

Assume the set function

$$s\colon \mathcal{M}' \to E \quad (\mathcal{M}' \supseteq \overline{\mathcal{K}})$$

in E^n and that $q \in E$.

(i) We say that q is the *derivative* of s at a point $\bar{p} \in E^n$ iff

$$q = \lim_{k \to \infty} \frac{sI_k}{mI_k}$$

for *all* sequences $\{I_k\} \subseteq \overline{\mathcal{K}}$, with $I_k \to \bar{p}$ (see Definition 1 in §10), Notation:

$$q = s'(\bar{p}) = \frac{d}{dm} s(\bar{p}).$$

If, in addition, $|q| < \infty$, we say that s is *differentiable* at $\bar{p}$.

If

$$q = \lim_{k \to \infty} \frac{sI_k}{mI_k}$$

for *at least one* such sequence $I_k \to \bar{p}$, we call q a *derivate* of s at $\bar{p}$ and write

$$q \sim Ds(\bar{p}).$$

If $s'(\bar{p})$ exists, it is the *unique* derivate of s at $\bar{p}$.

(ii) In case E is E^* or E^1, we admit *infinite* derivates and derivatives.

For *any* set function

$$s\colon \mathcal{M}' \to E^*$$

(measure or not) with

$$\mathcal{M}' \supseteq \overline{\mathcal{K}},$$

we also define

$$\underline{D}s(\bar{p}) \text{ and } \overline{D}s(\bar{p})$$

exactly as in Definition 3 of §10.

Equivalently, $\underline{D}s(\bar{p})$ is the *least* and $\overline{D}s(\bar{p})$ is the *largest* derivate of s at $\bar{p}$ (Problem 11 in §10). This shows that if $E = E^*$ or $E = E^1$, derivates exist at *every* $\bar{p}$.

Note 1. Hence $q = s'(\bar{p})$ in E^* iff

$$q = \underline{D}s(\bar{p}) = \overline{D}s(\bar{p}).$$

Note 2. We treat $\underline{D}s$, $\overline{D}s$, and s' as functions on *points* of E^n. Thus they are *point* functions, even though s is a *set* function.

The easy proofs of Theorems 1 and 2 (with $\overline{\mathcal{K}}$ and $\mathcal{M}' \supseteq \overline{\mathcal{K}}$ as above) are left to the reader.

Theorem 1. *If $s, t\colon \mathcal{M}' \to E$ are differentiable at $\bar{p}$, so are $s \pm t$ and ks for any scalar k. (If s, t are scalar valued, k may be a vector.) Moreover,*

$$(s \pm t)' = s' \pm t' \text{ and } (ks)' = ks' \text{ at } \bar{p}.$$

(See also Problem 7.)

Theorem 2. *A set function $s\colon \mathcal{M}' \to E^r\,(C^r)$ is differentiable at $\bar{p}$ iff its components $s_1, s_2, \ldots, s_r$ are; and then*

$$s' = (s'_1, \ldots, s'_r) = \sum_{i=1}^{r} \bar{e}_i s'_i \quad \text{at } \bar{p}.$$

In particular, for complex functions,

$$s' = s'_{\text{re}} + i \cdot s'_{\text{im}} \quad \text{at } \bar{p}.$$

The process described in Definition 1 will be called *Lebesgue differentiation* or $\overline{\mathcal{K}}$*-differentiation*, as opposed to "Ω*-differentiation*," defined next.[1]

---

[1] We follow some ideas by E. Munroe here.

Definition 2.

Let μ^* be a $\mathcal{G}$-regular (§5) outer measure in a metric space (S, ρ); recall that

$$\mathcal{G} = \{\text{all open sets in } S\}.$$

Let $\mu\colon \mathcal{M} \to E^*$ be the μ^*-induced (§6) measure in S.

A countable (two-indexed) set family

$$\Omega = \{U_n^i\} \subseteq \mathcal{M} \quad (i, n = 1, 2, \ldots)$$

is called a *network* in S (with respect to μ and ρ) iff

(i*) the space

$$S = \bigcup_{n=1}^{\infty} U_n^i \ (\textit{disjoint}), \quad i = 1, 2, \ldots,$$

with

$$0 < \mu U_n^i < \infty, \quad i, n = 1, 2, \ldots;^2$$

(ii*) each U_n^{i+1} is a subset of some U_r^i (the U_n^i *decrease* as i increases);

(iii*) for each $p \in S$, there is a sequence

$$\{I_k\} \subseteq \Omega,$$

with $I_k \to p$; that is,

$$p \in \bigcap_{k=1}^{\infty} I_k$$

and $dI_k \to 0$ (dI_k = diameter of I_k in (S, ρ)).

Now, given any set function

$$s\colon \mathcal{M}' \to E \quad (\mathcal{M}' \supseteq \Omega),$$

we define derivatives, derivates (also $\overline{D}s$ and $\underline{D}s$ if $E \subseteq E^*$), and differentiability exactly as in Definition 1, replacing $\overline{\mathcal{K}}$ by Ω, and Lebesgue measure m by μ.

Note that these derivates and derivatives depend not only on μ and ρ but also on the choice of Ω. To stress this, one might write $s'_{\mu\Omega}$ and $D_{\mu\Omega}s$ for s' and Ds, respectively. Mostly, however, no confusion is caused by simply writing s' and Ds (and we shall do so).

A network for E^n is suggested in the "hint" to Problem 2 of §2. See also Note 3.

---

[2] Thus for each *fixed* i, the U_n^i are *disjoint*. Also, μ is σ-finite, and $S \in \mathcal{M}$.

Theorems 1 and 2 carry over to Ω-differentiation, with the same proofs. We shall also need a substitute for the Vitali theorem (Theorem 1 of §10). It is quite simple.

Definition 3.

Let Ω be as in Definition 2. A set family $\mathcal{N} \subseteq \Omega$ is called an Ω-*covering* of $A \subseteq (S, \rho)$ iff

$$A \subseteq \bigcup \mathcal{N},$$

where $\bigcup \mathcal{N}$ is defined to be $\bigcup_{X \in \mathcal{N}} X$.

Theorem 3. *Let $\mathcal{N}$ be an Ω-covering of $A \subseteq S$. Then there is a disjoint sequence*

$$\{I_k\} \subseteq \mathcal{N}$$

with

$$A \subseteq \bigcup_k I_k$$

so that

$$\mu^*\left(A - \bigcup_k I_k\right) = 0$$

and

$$\mu^* A = \mu^*\left(A \cap \bigcup_k I_k\right).$$

Proof. As $\mathcal{N} \subseteq \Omega$, $\mathcal{N}$ consists of *some* of the U_n^i. For each i, let

$$\mathcal{N}^i = \{U_n^i \in \mathcal{N} \mid n = 1, \ldots\},$$

i.e., $\mathcal{N}^i$ consists of all $U_n^i \in \mathcal{N}$ with *that* particular index i.

Now, by Definition 2(i*)(ii*), any two U_n^i are either disjoint, or one contains the other. (Why?) Thus to construct $\{I_k\}$, start with all the (disjoint) $\mathcal{N}^1$-sets (if $\mathcal{N}^1 \neq \emptyset$). Then add those $U_n^2 \in \mathcal{N}^2$ that are *not* subsets of any set from $\mathcal{N}^1$ and hence are *disjoint* from such sets. Next, add those $U_n^3 \in \mathcal{N}^3$ that are not subsets of any set chosen from $\mathcal{N}^1$ or $\mathcal{N}^2$, and so on.

All U_n^i so chosen form a *disjoint* subfamily $\mathcal{K} \subseteq \mathcal{N}$ that covers all of A, as

$$A \subseteq \bigcup \mathcal{N} = \bigcup \mathcal{K}.$$

(Why?)

$\mathcal{K}$ is *countable* (as Ω is); so we can put it in a sequence $\{I_k\}$, with

$$A \subseteq \bigcup_k I_k \text{ (}\textit{disjoint}\text{)},$$

as required. $\square$

We can now prove our main result for $\overline{\mathcal{K}}$- and Ω-differentiation alike.

Theorem 4.

(i) *If* $s\colon \mathcal{M}' \to E^*\,(E^r, C^r)$ *is a generalized measure in* E^n, *finite on* $\overline{\mathcal{K}}$, *then* s *is differentiable a.e. on* E^n *(under Lebesgue measure* m*).*

(ii) *Similarly for* Ω*-differentiation in* (S, ρ), *provided* s *is finite on* Ω *and regular.*[3]

Proof. Via components and the Jordan decomposition (Theorem 4 of §11), all reduces to the case where s is a *measure* (≥ 0). Then the proof for $\overline{\mathcal{K}}$-differentiation is as in Lemmas 1 and 2 in §10. (Verify!)

For Ω-differentiation, the proof of Lemma 1 in §10 still works, with $\overline{\mathcal{K}}$-coverings replaced by Ω-coverings.

In the proof of Lemma 2, after choosing rationals $v > u$, we choose Q, $G \supseteq Q$, the Ω-covering

$$\mathcal{K} = \left\{ I \in \Omega \;\middle|\; I \subseteq G,\ \frac{sI}{\mu I} < u \right\}$$

of Q, and the sequence $\{I_k\} \subseteq \mathcal{K}$, as before. (In selecting G, we use the $\mathcal{G}$-regularity of μ^*; the I_k need not be cubes here, of course.)

Then, however, instead of forming the set Q_o, we use the regularity of s to select an open set $G' \in \mathcal{M}'$ with

$$G' \supseteq \bigcup_k I_k \supseteq Q$$

and

$$sG' - \varepsilon \leq s \bigcup I_k \leq \sum sI_k.$$

The set family

$$\mathcal{K}' = \left\{ I \in \Omega \;\middle|\; I \subseteq G',\ \frac{sI}{\mu I} > v \right\}$$

is then an Ω-covering of Q (why?); so we find a disjoint sequence $\{I_k'\} \subseteq \mathcal{K}'$ with

$$Q \subseteq \bigcup I_k' \subseteq G' \subseteq G$$

and obtain

$$\begin{aligned} u \cdot (\mu^* Q + \varepsilon) \geq u \cdot \mu G \geq u \cdot \sum_k \mu I_k \geq \sum_k sI_k \geq sG' - \varepsilon \geq \sum_k sI_k' - \varepsilon \\ \geq v \cdot \sum_k \mu I_k' - \varepsilon = v \cdot \mu \bigcup I_k' - \varepsilon \geq v \cdot \mu^* Q - \varepsilon. \end{aligned}$$

---

[3] A *signed* measure s is called regular iff s^+ and s^- are regular (Definition 4 in §7). A complex measure s is regular iff s_{re} and s_{im} are. Finally, $s\colon \mathcal{M}' \to E^r\,(C^r)$ is regular iff all its components s_i are.

Thus

$$(\forall\, \varepsilon > 0) \quad u \cdot (\mu^* Q + \varepsilon) \geq v \cdot \mu^* Q - \varepsilon.$$

The rest is as in Lemma 2 of §10. □

Note 3. If $\mu^* = m^*$, $\overline{\mathcal{K}}$-derivatives *equal* Ω-derivatives a.e. for a regular s (Problem 6). One may use Ω in E^n, *thus avoiding Theorem* 1 of §10 (Problem 13).

Problems on Differentiation of Set Functions

1. Complete the proofs of Theorems 1 to 4 in detail. Verify Note 1.

2. Verify that the hint for Problem 2 in §2 describes a network for E^n (see Note 3).

3. Show that the measure μ in Definition 2 is necessarily topological.
[Hint: Any $G \in \mathcal{G}$ is a countable union of Ω-sets. Why?]

4. (i) Show that the derivates of s at $\bar{p}$ form exactly the set $D'_{\bar{p}}$ of all cluster points of sequences sI_k/mI_k with $I_k \to \bar{p}$ and $\{I_k\} \subseteq \overline{\mathcal{K}}$. Do the same considering sequences $sI_k/\mu I_k$ with $I_k \to \bar{p}$ and $\{I_k\} \subseteq \Omega$.

(ii) Do Problem 11 in §10 for Ω-differentiation. Must s be regular here?

5. Verify that if

$$(\forall\, I \in \Omega) \quad \mu\overline{I} = \mu I^o,$$

then Theorem 4 holds for Ω-differentiation even if s is not regular.
[Hint: The proof of Lemma 2 of §10 holds *unchanged*.]

6. Prove Note 3 assuming that (i) s is regular, or (ii) $(\forall\, I \in \Omega)\ \mu\overline{I} = \mu I^o$ (see Problem 5).
[Hint: Imitate Problem 9(b) in §10 and the "Ω" part in the proof of Theorem 4.]

7. Prove for $\overline{\mathcal{K}}$- and Ω-differentiation that if

$$s = t \pm u \quad (s, t, u \colon \mathcal{M}' \to E^*)$$

and if u is differentiable at p, then $\overline{D}s = \overline{D}t \pm u'$ and $\underline{D}s = \underline{D}t \pm u'$ at p.

8. In Theorem 4 show that $\underline{D}s = \overline{D}s$ a.e. even if s is not finite on all of $\overline{\mathcal{K}}$ (Ω).
[Hint: For $s \geq 0$, Lemma 1 in §10, still holds. For *signed* measures, use Problem 7, noting that s^+ or s^- is *finite*, hence differentiable a.e.]

9. Prove that if f and $s = m_f$ are as in the proof of Theorem 3 in §10, then s and f are differentiable at the *same* points in E^1, and $s' = f'$ there.

[Hint: Use Note 1, Definition 1, and Chapter 5, §1, Problem 9, considering *one-sided* derivatives, f'_{+} and f'_{-}.]

10. Given a *universal* V-covering $\overline{\mathcal{K}}^*$ (see Problem 9 in §10), develop $\overline{\mathcal{K}}^*$-*differentiation* as in Definition 1, replacing $\overline{\mathcal{K}}$ by $\overline{\mathcal{K}}^*$ and writing s'^*, $\underline{D}^*s, \ldots$ for s', $\underline{D}s$, etc.

Extend Theorems 1–4 and Problem 7 to $\overline{\mathcal{K}}^*$-differentiation. Under the assumptions of Theorem 4, show that $s'^* = s'$ a.e. on E^n (use Problem 9 in §10).

11. Given a *normal* V-covering $\mathcal{K}^*$ of E^n (Problem 8 in §10), develop $\mathcal{K}^*$-*differentiation* along the lines of Problem 12 in §10 (admitting *normal* sequences $\{I_k\}$ only). Do the same questions as in Problem 10, for $\mathcal{K}^*$-differentiation.

12. Describe what changes if, in Problem 11, we drop the normality restriction on sequences $I_k \to \bar{p}$ (call it *strong* $\mathcal{K}^*$-differentiation; write $D^{**}s$, s'^{**}, etc.).

Show that

$$\underline{D}^{**}s \le \underline{D}^*s \le \overline{D}^*s \le \overline{D}^{**}s$$

on E^n, and so the existence of s'^{**} implies that of s'^*.

However the proof of Lemmas 1 and 2 in §10 fails for $\underline{D}^{**}s$ and $\overline{D}^{**}s$ (at what step?). So does the proof of Theorem 4. What about Theorems 1 and 2?

Chapter 8

Measurable Functions. Integration

§1. Elementary and Measurable Functions

From set functions, we now return to *point* functions

$$f \colon S \to (T, \rho')$$

whose domain D_f consists of *points* of a set S. The range space T will mostly be E, i.e., E^1, E^*, C, E^n, or another normed space. We assume $f(x) = 0$ unless defined otherwise. (In a general metric space T, we may take some fixed element q for 0.) Thus D_f is *all* of S, always.

We also adopt a convenient notation *for sets*:

$$\text{“}A(P)\text{” for “}\{x \in A \mid P(x)\}.\text{”}$$

Thus

$$\begin{aligned} A(f \neq a) &= \{x \in A \mid f(x) \neq a\}, \\ A(f = g) &= \{x \in A \mid f(x) = g(x)\}, \\ A(f > g) &= \{x \in A \mid f(x) > g(x)\}, \text{ etc.} \end{aligned}$$

Definition 1.

A *measurable space* is a set $S \neq \emptyset$ together with a set ring $\mathcal{M}$ of subsets of S, denoted $(S, \mathcal{M})$.

Henceforth, $(S, \mathcal{M})$ is fixed.

Definition 2.

An $\mathcal{M}$-*partition* of a set A is a countable set family $\mathcal{P} = \{A_i\}$ such that

$$A = \bigcup_i A_i \ (\textit{disjoint}),$$

with $A, A_i \in \mathcal{M}$.[1]

We briefly say “the partition $A = \bigcup A_i$.”

---

[1] $\mathcal{P}$ may be *finite*; it may even consist of A alone.

An $\mathcal{M}$-partition $\mathcal{P}' = \{B_{ik}\}$ is a *refinement* of $\mathcal{P} = \{A_i\}$ (or $\mathcal{P}'$ *refines* $\mathcal{P}$, or $\mathcal{P}'$ is *finer* than $\mathcal{P}$) iff

$$(\forall\, i) \quad A_i = \bigcup_k B_{ik};$$

i.e., each B_{ik} is contained in some A_i.

The *intersection* $\mathcal{P}' \Cap \mathcal{P}''$ of $\mathcal{P}' = \{A_i\}$ and $\mathcal{P}'' = \{B_k\}$ is understood to be the family of all sets of the form

$$A_i \cap B_k, \quad i, k = 1, 2, \ldots .$$

It is *an* $\mathcal{M}$*-partition that refines both* $\mathcal{P}'$ *and* $\mathcal{P}''$.

Definition 3.

A map (function) $f\colon S \to T$ is *elementary*, or $\mathcal{M}$*-elementary*, on a set $A \in \mathcal{M}$ iff there is an $\mathcal{M}$-partition $\mathcal{P} = \{A_i\}$ of A such that f is *constant* $(f = a_i)$ on each A_i.

If $\mathcal{P} = \{A_1, \ldots , A_q\}$ is *finite*, we say that f is *simple*, or $\mathcal{M}$*-simple*, on A.

If the A_i are *intervals* in E^n, we call f a *step function*; it is a *simple step function* if $\mathcal{P}$ is finite.[2]

The function values a_i are elements of T (possibly *vectors*). They may be infinite if $T = E^*$. Any simple map is also elementary, of course.

Definition 4.

A map $f\colon S \to (T, \rho')$ is said to be *measurable* (or $\mathcal{M}$*-measurable*) on a set A in $(S, \mathcal{M})$ iff

$$f = \lim_{m\to\infty} f_m \quad \text{(pointwise) on } A$$

for some sequence of functions $f_m\colon S \to T$, all elementary on A. (See Chapter 4, §12 for "pointwise.")

Note 1. This implies $A \in \mathcal{M}$, as follows from Definitions 2 and 3. (Why?)

Corollary 1. *If* $f\colon S \to (T, \rho')$ *is elementary on* A*, it is measurable on* A*.*

Proof. Set $f_m = f$, $m = 1, 2, \ldots$, in Definition 4. Then clearly $f_m \to f$ on A. $\square$

Corollary 2. *If* f *is simple, elementary, or measurable on* A *in* $(S, \mathcal{M})$*, it has the same property on any subset* $B \subseteq A$ *with* $B \in \mathcal{M}$*.*

---

[2] Only *simple step functions* are needed for a "limited approach." (One may proceed from here to §4, treating m as an additive *premeasure*.)

Proof. Let f be simple on A; so $f = a_i$ on A_i, $i = 1, 2, \ldots, n$, for some *finite* $\mathcal{M}$-partition, $A = \bigcup_{i=1}^{n} A_i$.

If $A \supseteq B \in \mathcal{M}$, then

$$\{B \cap A_i\}, \quad i = 1, 2, \ldots, n,$$

is a finite $\mathcal{M}$-partition *of* B (why?), and $f = a_i$ on $B \cap A_i$; so f is simple on B.

For elementary maps, use countable partitions.

Now let f be *measurable* on A, i.e.,

$$f = \lim_{m\to\infty} f_m$$

for some elementary maps f_m on A. As shown above, the f_m are elementary *on* B, too, and $f_m \to f$ on B; so f is measurable on B. $\square$

Corollary 3. *If f is elementary or measurable on each of the (countably many) sets A_n in $(S, \mathcal{M})$, it has the same property on their union $A = \bigcup_n A_n$.*

Proof. Let f be elementary on each A_n (so $A_n \in \mathcal{M}$ by Note 1).

By Corollary 1 of Chapter 7, §1,

$$A = \bigcup A_n = \bigcup B_n$$

for some *disjoint* sets $B_n \subseteq A_n$ $(B_n \in \mathcal{M})$.

By Corollary 2, f is elementary on each B_n; i.e., constant on sets of some $\mathcal{M}$-partition $\{B_{ni}\}$ *of* B_i.

All B_{ni} *combined* (for all n and all i) form an $\mathcal{M}$-partition *of* A,

$$A = \bigcup_n B_n = \bigcup_{n,i} B_{ni}.$$

As f is constant on *each* B_{ni}, it is elementary on A.

For *measurable* functions f, slightly *modify* the method used in Corollary 2. $\square$

Corollary 4. *If $f\colon S \to (T, \rho')$ is measurable on A in $(S, \mathcal{M})$, so is the composite map $g \circ f$, provided $g\colon T \to (U, \rho'')$ is relatively continuous on $f[A]$.*

Proof. By assumption,

$$f = \lim_{m\to\infty} f_m \text{ (pointwise)}$$

for some elementary maps f_m on A.

Hence by the continuity of g,

$$g(f_m(x)) \to g(f(x)),$$

i.e., $g \circ f_m \to g \circ f$ (pointwise) on A.

Moreover, all $g \circ f_m$ are elementary on A (for $g \circ f_m$ is constant on any partition set, if f_m is).

Thus $g \circ f$ is measurable on A, as claimed. $\square$

Theorem 1. *If the maps* $f, g, h\colon S \to E^1\,(C)$ *are simple, elementary, or measurable on* A *in* $(S, \mathcal{M})$, *so are* $f \pm g$, fh, $|f|^a$ (*for real* $a \neq 0$) *and* f/h (*if* $h \neq 0$ *on* A).

Similarly for vector-valued f *and* g *and* scalar-valued h.

Proof. First, let f and g be elementary on A. Then there are two $\mathcal{M}$-partitions,

$$A = \bigcup A_i = \bigcup B_k,$$

such that $f = a_i$ on A_i and $g = b_k$ on B_k, say.

The sets $A_i \cap B_k$ (for *all* i and k) then form a new $\mathcal{M}$-partition of A (why?), such that *both* f and g are constant on each $A_i \cap B_k$ (why?); hence so is $f \pm g$.

Thus $f \pm g$ is elementary on A. Similarly for simple functions.

Next, let f and g be *measurable* on A; so

$$f = \lim f_m \text{ and } g = \lim g_m \text{ (pointwise) on } A$$

for some elementary maps f_m, g_m.

By what was shown above, $f_m \pm g_m$ is elementary for each m. Also,

$$f_m \pm g_m \to f \pm g \text{ (pointwise) on } A.$$

Thus $f \pm g$ is measurable on A.

The rest of the theorem follows quite similarly. $\square$

If the range space is E^n (or C^n), then f has n real (complex) components $f_1, \ldots, f_n$, as in Chapter 4, §3 (Part II). This yields the following theorem.

Theorem 2. *A function* $f\colon S \to E^n\,(C^n)$ *is simple, elementary, or measurable on a set* A *in* $(S, \mathcal{M})$ *iff all its* n *component functions* $f_1, f_2, \ldots, f_n$ *are.*

Proof. For simplicity, consider $f\colon S \to E^2$, $f = (f_1, f_2)$.

If f_1 and f_2 are simple or elementary on A then (exactly as in Theorem 1), one can achieve that *both* are constant on sets $A_i \cap B_k$ of *one and the same* $\mathcal{M}$-partition of A. Hence $f = (f_1, f_2)$, too, is constant on each $A_i \cap B_k$, as required.

Conversely, let

$$f = \bar{c}_i = (a_i, b_i) \text{ on } C_i$$

for some $\mathcal{M}$-partition

$$A = \bigcup C_i.$$

Then by definition, $f_1 = a_i$ and $f_2 = b_i$ on C_i; so both are elementary (or simple) on A.

In the general case (E^n or C^n), the proof is analogous.

For *measurable* functions, the proof reduces to limits of *elementary maps* (using Theorem 2 of Chapter 3, §15). The details are left to the reader. $\square$

Note 2. As $C = E^2$, a *complex function* $f : S \to C$ *is simple, elementary, or measurable on* A *iff its real and imaginary parts are.*

By Definition 4, a measurable function is a *pointwise* limit of elementary maps. However, if $\mathcal{M}$ is a σ*-ring*, one can make the limit *uniform*. Indeed, we have the following theorem.

Theorem 3. *If* $\mathcal{M}$ *is a* σ*-ring, and* $f\colon S \to (T, \rho')$ *is* $\mathcal{M}$*-measurable on* A*, then*

$$f = \lim_{m\to\infty} g_m \ (\textit{uniformly}) \ \textit{on } A$$

for some finite elementary maps g_m.

Thus given $\varepsilon > 0$, *there is a finite elementary map* g *such that* $\rho'(f, g) < \varepsilon$ *on* A.[3,4]

The proof will be given in §2 for $T = E^*$. The general case is sketched in Problem 7 of §2. Meanwhile, we take the theorem for granted and use it below.

Theorem 4. *If* $\mathcal{M}$ *is a* σ*-ring in* S*, if*

$$f_m \to f \ (\textit{pointwise}) \ \textit{on } A$$

($f_m\colon S \to (T, \rho')$), *and if all* f_m *are* $\mathcal{M}$*-measurable on* A*, so also is* f.[4]

Briefly: *A pointwise limit of measurable maps is measurable* (unlike *continuous* maps; cf. Chapter 4, §12).

Proof. By the second clause of Theorem 3, each f_m is *uniformly* approximated by some *elementary* map g_m on A, so that, taking $\varepsilon = 1/m$, $m = 1, 2, \ldots,$

$$\rho'(f_m(x), g_m(x)) < \frac{1}{m} \quad \text{for all } x \in A \text{ and all } m. \tag{1}$$

Fixing such a g_m for each m, we show that $g_m \to f$ (*pointwise*) on A, as required in Definition 4.

Indeed, fix any $x \in A$. By assumption, $f_m(x) \to f(x)$. Hence, given $\delta > 0$,

$$(\exists\, k)\ (\forall\, m > k) \quad \rho'(f(x), f_m(x)) < \delta.$$

Take k so large that, in addition,

$$(\forall\, m > k) \quad \frac{1}{m} < \delta.$$

---

[3] We briefly write $\rho'(f, g)$ for $\sup_{x\in S} \rho'(f(x), g(x))$.

[4] The theorem holds also for $T = E^*$, with ρ' as in Problem 5 of Chapter 3, §11.

Then by the triangle law and by (1), we obtain for $m > k$ that

$$\begin{aligned}\rho'(f(x), g_m(x)) &\le \rho'(f(x), f_m(x)) + \rho'(f_m(x), g_m(x)) \\ &< \delta + \frac{1}{m} < 2\delta.\end{aligned}$$

As δ is arbitrary, this implies $\rho'(f(x), g_m(x)) \to 0$, i.e., $g_m(x) \to f(x)$ for any (fixed) $x \in A$, thus proving the measurability of f. □

Note 3. If

$$\mathcal{M} = \mathcal{B} \ (= \text{Borel field in } S),$$

we often say "*Borel measurable*" for $\mathcal{M}$-measurable. If

$$\mathcal{M} = \{\text{Lebesgue measurable sets in } E^n\},$$

we say "*Lebesgue* (L) *measurable*" instead. Similarly for "*Lebesgue–Stieltjes* (LS) *measurable*."

Problems on Measurable and Elementary Functions in $(S, \mathcal{M})$

1. Fill in all proof details in Corollaries 2 and 3 and Theorems 1 and 2.

2. Show that $\mathcal{P}' \sqcap \mathcal{P}''$ is as stated at the end of Definition 2.

3. Given $A \subseteq S$ and $f, f_m \colon S \to (T, \rho')$, $m = 1, 2, \ldots$, let

$$H = A(f_m \to f)$$

and

$$A_{mn} = A\Big(\rho'(f_m, f) < \frac{1}{n}\Big).$$

Prove that

(i) $H = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} A_{mn}$;

(ii) $H \in \mathcal{M}$ if all A_{mn} are in $\mathcal{M}$ and $\mathcal{M}$ is a *σ-ring*.

[Hint: $x \in H$ iff

$$(\forall n)\ (\exists k)\ (\forall m \ge k) \quad x \in A_{mn}.$$

Why?]

3′. Do Problem 3 for $T = E^*$ and $f = \pm\infty$ on H.
[Hint: If $f = +\infty$, $A_{mn} = A(f_m > n)$.]

⇒4. Let $f \colon S \to T$ be $\mathcal{M}$-elementary on A, with $\mathcal{M}$ a *σ-ring* in S. Show the following.

(i) $A(f = a) \in \mathcal{M}$, $A(f \neq a) \in \mathcal{M}$.

(ii) If $T = E^*$, then

$$A(f < a),\ A(f \geq a),\ A(f > a), \text{ and } A(f \geq a)$$

are in $\mathcal{M}$, too.

(iii) $(\forall\, B \subseteq T)\ A \cap f^{-1}[B] \in \mathcal{M}$.

[Hint: If

$$A = \bigcup_{i=1}^{\infty} A_i$$

and $f = a_i$ on A_i, then $A(f = a)$ is the *countable* union of those A_i for which $a_i = a$.]

5. Do Problem 4(i) for *measurable* f.

[Hint: If $f = \lim f_m$ for elementary maps f_m, then

$$H = A(f = a) = A(f_m \to a).$$

Express H as in Problem 3, with

$$A_{mn} = A\Bigl(h_m < \frac{1}{n}\Bigr),$$

where $h_m = \rho'(f_m, a)$ is *elementary*. (Why?) Then use Problems 4(ii) and 3(ii).]

⇒6. Given $f, g\colon S \to (T, \rho')$, let $h = \rho'(f, g)$, i.e.,

$$h(x) = \rho'(f(x), g(x)).$$

Prove that if f and g are elementary, simple, or measurable on A, so is h.

[Hint: Argue as in Theorem 1. Use Theorem 4 in Chapter 3, §15.]

⇒7. A set $B \subseteq (T, \rho')$ is called *separable* (in T) iff $B \subseteq \overline{D}$ (closure of D) for a *countable* set $D \subseteq T$.

Prove that if $f\colon S \to T$ is $\mathcal{M}$-measurable on A, then $f[A]$ is separable in T.

[Hint: $f = \lim f_m$ for elementary maps f_m; say,

$$f_m = a_{mi} \text{ on } A_{mi} \in \mathcal{M}, \quad i = 1, 2, \dots.$$

Let D consist of *all* a_{mi} $(m, i = 1, 2, \dots)$; so D is countable (why?) and $D \subseteq T$. Verify that

$$(\forall\, y \in f[A])\ (\exists\, x \in A) \quad y = f(x) = \lim f_m(x),$$

with $f_m(x) \in D$. Hence

$$(\forall\, y \in f[A]) \quad y \in \overline{D},$$

by Theorem 3 of Chapter 3, §16.]

⇒8. Continuing Problem 7, prove that if $B \subseteq \overline{D}$ and $D = \{q_1, q_2, \dots\}$, then

$$(\forall\, n) \quad B \subseteq \bigcup_{i=1}^{\infty} G_{q_i}\Bigl(\frac{1}{n}\Bigr).$$

[Hint: If $p \in B \subseteq \overline{D}$, *any* $G_p\left(\frac{1}{n}\right)$ contains some $q_i \in D$; so

$$\rho'(p, q_i) < \frac{1}{n}, \text{ or } p \in G_{q_i}\left(\frac{1}{n}\right).$$

Thus

$$(\forall\, p \in B) \quad p \in \bigcup_{i=1}^{\infty} G_{q_i}\left(\frac{1}{n}\right).]$$

9. Prove Corollaries 2 and 3 and Theorems 1 and 2, assuming that $\mathcal{M}$ is a *semiring* only.

10. Do Problem 4 for $\mathcal{M}$-*simple* maps, assuming that $\mathcal{M}$ is a *ring* only.

§2. Measurability of Extended-Real Functions

Henceforth we presuppose a measurable space $(S, \mathcal{M})$, where $\mathcal{M}$ is a σ-*ring* in S. Our aim is to prove the following basic theorem, which is often used as a *definition*, for *extended-real* functions $f\colon S \to E^*$.

Theorem 1. *A function $f\colon S \to E^*$ is measurable on a set $A \in \mathcal{M}$ iff it satisfies one of the following equivalent conditions* (*hence all of them*):

$$\text{(i}^*\text{)} \ (\forall\, a \in E^*)\ A(f > a) \in \mathcal{M}; \qquad \text{(ii}^*\text{)} \ (\forall\, a \in E^*)\ A(f \geq a) \in \mathcal{M};$$
$$\text{(iii}^*\text{)} \ (\forall\, a \in E^*)\ A(f < a) \in \mathcal{M}; \qquad \text{(iv}^*\text{)} \ (\forall\, a \in E^*)\ A(f \leq a) \in \mathcal{M}.$$

We first prove the *equivalence* of these conditions by showing that (i*) ⇒ (ii*) ⇒ (iii*) ⇒ (iv*) ⇒ (i*), closing the "circle."

(i*) ⇒ (ii*). Assume (i*). If $a = -\infty$,

$$A(f \geq a) = A \in \mathcal{M}$$

by assumption. If $a = +\infty$,

$$A(f \geq a) = A(f = \infty) = \bigcap_{n=1}^{\infty} A(f > n) \in \mathcal{M}$$

by (i*). And if $a \in E^1$,

$$A(f \geq a) = \bigcap_{n=1}^{\infty} A\left(f > a - \frac{1}{n}\right).$$

(Verify!) By (i*),

$$A\left(f > a - \frac{1}{n}\right) \in \mathcal{M};$$

so $A(f \geq a) \in \mathcal{M}$ (a σ-ring!).

(ii*) ⇒ (iii*). For (ii*) and $A \in \mathcal{M}$ imply

$$A(f < a) = A - A(f \geq a) \in \mathcal{M}.$$

(iii*) ⇒ (iv*). If $a \in E^1$,

$$A(f \leq a) = \bigcap_{n=1}^{\infty} A\Big(f < a + \frac{1}{n}\Big) \in \mathcal{M}.$$

What if $a = \pm\infty$?

(iv*) ⇒ (i*). Indeed, (iv*) and $A \in \mathcal{M}$ imply

$$A(f > a) = A - A(f \leq a) \in \mathcal{M}.$$

Thus, indeed, each of (i*) to (iv*) implies the others. To finish, we need two lemmas that are of interest in their own right.

Lemma 1. *If the maps* $f_m\colon S \to E^*$ $(m = 1, 2, \ldots)$ *satisfy conditions* (i*)–(iv*), *so also do the functions*

$$\sup f_m, \ \inf f_m, \ \overline{\lim}\, f_m, \ \textit{and} \ \underline{\lim}\, f_m,$$

defined pointwise, *i.e.*,

$$(\sup f_m)(x) = \sup f_m(x),$$

and similarly for the others.

Proof. Let $f = \sup f_m$. Then

$$A(f \leq a) = \bigcap_{m=1}^{\infty} A(f_m \leq a). \quad \text{(Why?)}$$

But by assumption,

$$A(f_m \leq a) \in \mathcal{M}$$

(f_m satisfies (iv*)). Hence $A(f \leq a) \in \mathcal{M}$ (for $\mathcal{M}$ is a σ-*ring*).

Thus $\sup f_m$ satisfies (i*)–(iv*).

So does $\inf f_m$; for

$$A(\inf f_m \geq a) = \bigcap_{m=1}^{\infty} A(f_m \geq a) \in \mathcal{M}.$$

(Explain!)

So also do $\underline{\lim}\, f_m$ and $\overline{\lim}\, f_m$; for by definition,

$$\underline{\lim}\, f_m = \sup_k g_k,$$

where

$$g_k = \inf_{m \geq k} f_m$$

satisfies (i*)–(iv*), as was shown above; hence so does $\sup g_k = \underline{\lim}\, f_m$.

Similarly for $\overline{\lim}\, f_m$. $\square$

Lemma 2. *If f satisfies* (i*)–(iv*), *then*

$$f = \lim_{m\to\infty} f_m \ (\textit{uniformly}) \ \textit{on } A$$

for some sequence of finite functions f_m, all $\mathcal{M}$-elementary on A.

Moreover, if $f \geq 0$ on A, the f_m can be made nonnegative, with $\{f_m\}\uparrow$ on A.

Proof. Let $H = A(f = +\infty)$, $K = A(f = -\infty)$, and

$$A_{mk} = A\Big(\frac{k-1}{2^m} \leq f < \frac{k}{2^m}\Big)$$

for $m = 1, 2, \ldots$ and $k = 0, \pm 1, \pm 2, \ldots, \pm n, \ldots$.

By (i*)–(iv*),

$$H = A(f = +\infty) = A(f \geq +\infty) \in \mathcal{M},$$

$K \in \mathcal{M}$, and

$$A_{mk} = A\Big(f \leq \frac{k-1}{2^m}\Big) \cap A\Big(f < \frac{k}{2^m}\Big) \in \mathcal{M}.$$

Now define

$$(\forall\, m) \quad f_m = \frac{k-1}{2^m} \text{ on } A_{mk},$$

$f_m = m$ on H, and $f_m = -m$ on K. Then each f_m is finite and *elementary on* A since

$$(\forall\, m) \quad A = H \cup K \cup \bigcup_{k=-\infty}^{\infty} A_{mk} \ (\textit{disjoint})$$

and f_m is constant on H, K, and A_{mk}.

We now show that $f_m \to f$ (uniformly) on H, K, and

$$J = \bigcup_{k=-\infty}^{\infty} A_{mk},$$

hence on A.

Indeed, on H we have

$$\lim f_m = \lim m = +\infty = f,$$

and the limit is *uniform* since the f_m are constant on H.

Similarly,

$$f_m = -m \to -\infty = f \text{ on } K.$$

Finally, on A_{mk} we have

$$(k-1)\, 2^{-m} \leq f < k\, 2^{-m}$$

and $f_m = (k-1)\,2^{-m}$; hence

$$|f_m - f| < k\,2^{-m} - (k-1)\,2^{-m} = 2^{-m}.$$

Thus

$$|f_m - f| < 2^{-m} \to 0$$

on *each* A_{mk}, hence on

$$J = \bigcup_{k=-\infty}^{\infty} A_{mk}.$$

By Theorem 1 of Chapter 4, §12, it follows that $f_m \to f$ (uniformly) on J. Thus, indeed, $f_m \to f$ (uniformly) on A.

If, further, $f \geq 0$ on A, then $K = \emptyset$ and $A_{mk} = \emptyset$ for $k \leq 0$. Moreover, on passage from m to $m+1$, each A_{mk} $(k > 0)$ splits into *two* sets. On one, $f_{m+1} = f_m$; on the other, $f_{m+1} > f_m$. (Why?)

Thus $0 \leq f_m \nearrow f$ (uniformly) on A, and all is proved. $\square$

Proof of Theorem 1. If f is measurable on A, then by definition, $f = \lim f_m$ (pointwise) for some elementary maps f_m on A.

By Problem 4(ii) in §1, *all* f_m *satisfy* (i*)–(iv*). Thus so does f by Lemma 1, for here $f = \lim f_m = \overline{\lim} f_m$.

The converse follows by Lemma 2. This completes the proof. $\square$

Note 1. Lemmas 1 and 2 prove Theorems 3 and 4 of §1, *for* $f\colon S \to E^*$. By using also Theorem 2 in §1, one easily extends this to $f\colon S \to E^n$ (C^n). Verify!

Corollary 1. *If* $f\colon S \to E^*$ *is measurable on* A, *then*

$$(\forall\, a \in E^*) \quad A(f = a) \in \mathcal{M} \text{ and } A(f \neq a) \in \mathcal{M}.$$

Indeed,

$$A(f = a) = A(f \geq a) \cap A(f \leq a) \in \mathcal{M}$$

and

$$A(f \neq a) = A - A(f = a) \in \mathcal{M}.$$

Corollary 2. *If* $f\colon S \to (T, \rho')$ *is measurable on* A *in* $(S, \mathcal{M})$, *then*

$$A \cap f^{-1}[G] \in \mathcal{M}$$

for every globe $G = G_q(\delta)$ *in* (T, ρ').

Proof. Define $h\colon S \to E^1$ by

$$h(x) = \rho'(f(x), q).$$

Then h is measurable on A by Problem 6 in §1. Thus by Theorem 1,

$$A(h < \delta) \in \mathcal{M}.$$

But as is easily seen,

$$A(h < \delta) = \{x \in A \mid \rho'(f(x), q) < \delta\} = A \cap f^{-1}[G_q(\delta)].$$

Hence the result. □

Definition.

Given $f, g : S \to E^*$, we define the maps $f \vee g$ and $f \wedge g$ on S by

$$(f \vee g)(x) = \max\{f(x), g(x)\}$$

and

$$(f \wedge g)(x) = \min\{f(x), g(x)\};$$

similarly for $f \vee g \vee h$, $f \wedge g \wedge h$, etc.

We also set

$$f^+ = f \vee 0 \text{ and } f^- = -f \vee 0.$$

Clearly, $f^+ \geq 0$ and $f^- \geq 0$ on S. Also, $f = f^+ - f^-$ and $|f| = f^+ + f^-$. (Why?) We now obtain the following theorem.

Theorem 2. *If the functions $f, g\colon S \to E^*$ are simple, elementary, or measurable on A, so also are $f \pm g$, fg, $f \vee g$, $f \wedge g$, f^+, f^-, and $|f|^a$ $(a \neq 0)$.*

Proof. If f and g are *finite*, this follows by Theorem 1 of §1 on verifying that

$$f \vee g = \frac{1}{2}(f + g + |f - g|)$$

and

$$f \wedge g = \frac{1}{2}(f + g - |f - g|)$$

on S. (Check it!)

Otherwise, consider

$$A(f = +\infty),\ A(f = -\infty),\ A(g = +\infty), \text{ and } A(g = -\infty).$$

By Theorem 1, these are $\mathcal{M}$-sets; hence so is their union U.

On each of them $f \vee g$ and $f \wedge g$ *equal f or g*; so by Corollary 3 in §1, $f \vee g$ and $f \wedge g$ have the desired properties *on U*. So also have $f^+ = f \vee 0$ and $f^- = -f \vee 0$ (take $g = 0$).

We claim that the maps $f \pm g$ and fg are *simple* (hence elementary and measurable) *on each of the four sets mentioned above*, hence on U.

For example, on $A(f = +\infty)$,

$$f \pm g = +\infty \ (\textit{constant})$$

by our conventions (2^*) in Chapter 4, §4. For fg, split $A(f = +\infty)$ into three sets A_1, A_2, $A_3 \in \mathcal{M}$, with $g > 0$ on A_1, $g < 0$ on A_2, and $g = 0$ on A_3; so $fg = +\infty$ on A_1, $fg = -\infty$ on A_2, and $fg = 0$ on A_3. Hence fg is *simple* on $A(f = +\infty)$.

For $|f|^a$, use $U = A(|f| = \infty)$. Again, the theorem holds *on* U, and also *on* $A-U$, since f and g are *finite* on $A-U \in \mathcal{M}$. Thus it holds on $A = (A-U) \cup U$, by Corollary 3 in §1. $\square$

Note 2. Induction extends Theorem 2 to any finite number of functions.

Note 3. Combining Theorem 2 with $f = f^+ - f^-$, we see that $f : S \to E^*$ is simple (elementary, measurable) iff f^+ and f^- are. We also obtain the following result.

Theorem 3. *If the functions* $f, g \colon S \to E*$ *are measurable on* $A \in \mathcal{M}$, *then* $A(f \geq g) \in \mathcal{M}$, $A(f < g) \in \mathcal{M}$, $A(f = g) \in \mathcal{M}$, *and* $A(f \neq g) \in \mathcal{M}$.

(See Problem 4 below.)

Further Problems on Measurable Functions in $(S, \mathcal{M})$

1. In Theorem 1, give the details in proving the equivalence of (i^*)–(iv^*).

2. Prove Note 1.

2′. Prove that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

3. Complete the proof of Theorem 2, in detail.

⇒4. Prove Theorem 3.

[Hint: By our conventions, $A(f \geq g) = A(f - g \geq 0)$ even if g or f is $\pm\infty$ for some $x \in A$. (Verify all cases!) By Theorems 1 and 2, $A(f - g \geq 0) \in \mathcal{M}$; so $A(f \geq g) \in \mathcal{M}$, and $A(f < g) = A - A(f \geq g) \in \mathcal{M}$. Proceed.]

5. Show that the measurability of $|f|$ does *not* imply that of f.

[Hint: Let $f = 1$ on Q and $f = -1$ on $A - Q$ for some $Q \notin \mathcal{M}$ ($Q \subset A$); e.g., use Q of Problem 6 in Chapter 7, §8.]

⇒6. Show that a function $f \geq 0$ is measurable on A iff $f_m \nearrow f$ (pointwise) on A for some finite *simple* maps $f_m \geq 0$, $\{f_m\}\!\uparrow$.

[Hint: Modify the proof of Lemma 2, setting $H_m = A(f \geq m)$ and $f_m = m$ on H_m, and defining the A_{mk} for $1 \leq k \leq m2^m$ only.]

⇒7. Prove Theorem 3 in §1.

[Outline: By Problems 7 and 8 in §1, there are $q_i \in T$ such that

$$(\forall n) \quad f[A] \subseteq \bigcup_{i=1}^{\infty} G_{q_i}\Big(\frac{1}{n}\Big).$$

Set

$$A_{ni} = A \cap f^{-1}\Big[G_{q_i}\Big(\frac{1}{n}\Big)\Big] \in \mathcal{M}$$

by Corollary 2; so $\rho'(f(x), q_i) < \frac{1}{n}$ on A_{ni}.

By Corollary 1 in Chapter 7, §1,

$$A = \bigcup_{i=1}^{\infty} A_{ni} = \bigcup_{i=1}^{\infty} B_{ni} \ (\textit{disjoint})$$

for some sets $B_{ni} \in \mathcal{M}$, $B_{ni} \subseteq A_{ni}$. Now define $g_n = q_i$ on B_{ni}; so $\rho'(f, g_n) < \frac{1}{n}$ on each B_{ni}, hence *on* A. By Theorem 1 in Chapter 4, §12, $g_n \to f$ (uniformly) on A.]

⇒8. Prove that $f\colon S \to E^1$ is $\mathcal{M}$-measurable on A iff $A \cap f^{-1}[B] \in \mathcal{M}$ for every *Borel* set B (equivalently, for every *open* set B) in E^1. (In the case $f\colon S \to E^*$, add: "and for $B = \{\pm\infty\}$.")
[Outline: Let

$$\mathcal{R} = \{X \subseteq E^1 \mid A \cap f^{-1}[X] \in \mathcal{M}\}.$$

Show that $\mathcal{R}$ is a σ-ring in E^1.

Now, by Theorem 1, if f is measurable on A, $\mathcal{R}$ contains all *open intervals*; for

$$A \cap f^{-1}[(a, b)] = A(f > a) \cap A(f < b).$$

Then by Lemma 2 of Chapter 7, §2, $\mathcal{R} \supseteq \mathcal{G}$, hence $\mathcal{R} \supseteq \mathcal{B}$. (Why?)

Conversely, if so,

$$(a, \infty) \in \mathcal{R} \Rightarrow A \cap f^{-1}[(a, \infty)] \in \mathcal{M} \Rightarrow A(f > a) \in \mathcal{M}.]$$

⇒9. Do Problem 8 for $f\colon S \to E^n$.
[Hint: If $f = (f_1, \ldots, f_n)$ and $B = (\bar{a}, \bar{b}) \subset E^n$, with $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$, show that

$$f^{-1}[B] = \bigcap_{k=1}^{n} f_k^{-1}[(a_k, b_k)].$$

Apply Problem 8 to each $f_k\colon S \to E^1$ and use Theorem 2 in §1. Proceed as in Problem 8.]

10. Do Problem 8 for $f\colon S \to C^n$, treating C^n as E^{2n}.

11. Prove that $f\colon S \to (T, \rho')$ is measurable on A in $(S, \mathcal{M})$ iff

(i) $A \cap f^{-1}[G] \in \mathcal{M}$ for every open globe $G \subseteq T$, and

(ii) $f[A]$ is separable in T (Problem 7 in §1).

[Hint: If so, proceed as in Problem 7 (*without* assuming measurability of f) to show that $f = \lim g_n$ for some elementary maps g_n on A. For the converse, use Problem 7 in §1 and Corollary 2 in §2.]

12. (i) Show that if *all* of T is separable (Problem 7 in §1), there is a sequence of globes $G_k \subseteq T$ such that each nonempty open set $B \subseteq T$ is the union of some of these G_k.

(ii) Show that E^n and C^n are separable.

[Hints: (i) Use the $G_{q_i}(\frac{1}{n})$ of Problem 8 in §1, putting them in *one* sequence. (ii) Take $D = R^n \subset E^n$ in Problem 7 of §1.]

13. Do Problem 11 with "globe $G \subseteq T$" replaced by "Borel set $B \subseteq T$."
[Hints: Treat f as $f\colon A \to T'$, $T' = f[A]$, noting that

$$A \cap f^{-1}[B] = A \cap f^{-1}[B \cap T'].$$

By Problem 12, if $B \neq \emptyset$ is open in T, then $B \cap T'$ is a countable union of "globes" $G_q \cap T'$ in (T', ρ'); see Theorem 4 in Chapter 3, §12. Proceed as in Problem 8, replacing E^1 by T.]

14. A map $g\colon (T, \rho') \to (U, \rho'')$ is said to be of *Baire class* 0 ($g \in \mathbf{B}_0$) on $D \subseteq T$ iff g is relatively continuous on D. Inductively, g is of *Baire class* n ($g \in \mathbf{B}_n$, $n \geq 1$) iff $g = \lim g_m$ (pointwise) on D for some maps $g_m \in \mathbf{B}_{n-1}$. Show by induction that Corollary 4 in §1 holds also if $g \in \mathbf{B}_n$ on $f[A]$ for some n.

§3. Measurable Functions in $(S, \mathcal{M}, m)$

I. Henceforth we shall presuppose not just a *measurable space* (§1) but a *measure space* $(S, \mathcal{M}, m)$, where $m\colon \mathcal{M} \to E^*$ is a measure on a σ-ring $\mathcal{M} \subseteq 2^S$.

We saw in Chapter 7 that one could often neglect sets of Lebesgue measure zero on E^n—if a property held everywhere except on a set of Lebesgue measure zero, we said it held "almost everywhere." The following definition generalizes this usage.

Definition 1.

We say that a property $P(x)$ holds for *almost all* $x \in A$ (with respect to the measure m) or *almost everywhere* (a.e.(m)) on A iff it holds on $A - Q$ for some $Q \in \mathcal{M}$ with $mQ = 0$.

Thus we write

$$f_n \to f \ (a.e.) \text{ or } f = \lim f_n \ (a.e.(m)) \text{ on } A$$

iff $f_n \to f$ (pointwise) on $A - Q$, $mQ = 0$. Of course, "*pointwise*" implies "*a.e.*" (take $Q = \emptyset$), but the converse fails.

Definition 2.

We say that $f\colon S \to (T, \rho')$ is *almost measurable* on A iff $A \in \mathcal{M}$ and f is $\mathcal{M}$-measurable on $A - Q$, $mQ = 0$.

We then also say that f is *m-measurable* (m being the measure involved) as opposed to $\mathcal{M}$-measurable.

Observe that we may assume $Q \subseteq A$ here (replace Q by $A \cap Q$).

***Note 1.** If m is a *generalized measure* (Chapter 7, §11), replace $mQ = 0$ by $v_m Q = 0$ (v_m = *total variation* of m) in Definitions 1 and 2 and in the following proofs.

Corollary 1. *If the functions*

$$f_n\colon S \to (T, \rho'), \quad n = 1, 2, \ldots,$$

are m-measurable on A, and if

$$f_n \to f \ (a.e.(m))$$

on A, then f is m-measurable on A.

Proof. By assumption, $f_n \to f$ (pointwise) on $A - Q_0$, $mQ_0 = 0$. Also, f_n is $\mathcal{M}$-measurable on

$$A - Q_n, \ mQ_n = 0, \quad n = 1, 2, \ldots .$$

(The Q_n *need not be the same*.)

Let

$$Q = \bigcup_{n=0}^{\infty} Q_n;$$

so

$$mQ \le \sum_{n=0}^{\infty} mQ_n = 0.$$

By Corollary 2 in §1, *all* f_n are $\mathcal{M}$-measurable *on* $A - Q$ (why?), and $f_n \to f$ (pointwise) on $A - Q$, as $A - Q \subseteq A - Q_0$.

Thus by Theorem 4 in §1, f is $\mathcal{M}$-measurable on $A - Q$. As $mQ = 0$, this is the desired result. □

Corollary 2. *If $f = g$ (a.e. (m)) on A and f is m-measurable on A, so is g.*

Proof. By assumption, $f = g$ on $A - Q_1$ and f is $\mathcal{M}$-measurable on $A - Q_2$, with $mQ_1 = mQ_2 = 0$.

Let $Q = Q_1 \cup Q_2$. Then $mQ = 0$ and $g = f$ on $A - Q$. (Why?)

By Corollary 2 of §1, f is $\mathcal{M}$-measurable *on* $A - Q$. Hence so is g, as claimed. □

Corollary 3. *If $f\colon S \to (T, \rho')$ is m-measurable on A, then*

$$f = \lim_{n\to\infty} f_n \ (uniformly) \ on \ A - Q \ (mQ = 0),$$

for some maps f_n, all elementary on $A - Q$.

(Compare Corollary 3 with Theorem 3 in §1).

Quite similarly all other propositions of §1 carry over to *almost* measurable (i.e., *m-measurable*) functions. Note, however, that the term "measurable" in §§1 and 2 always meant "$\mathcal{M}$-measurable." This *implies* m-measurability (take $Q = \emptyset$), but the converse fails. (See Note 2, however.)

We still obtain the following result.

Corollary 4. *If the functions*

$$f_n\colon S \to E^* \quad (n = 1, 2, \ldots)$$

are m-measurable on a set A, so also are

$$\sup f_n,\ \inf f_n,\ \overline{\lim}\, f_n,\ \textit{and}\ \underline{\lim}\, f_n.$$

(Use Lemma 1 of §2).

Similarly, Theorem 2 in §2 carries over to m-measurable functions.

Note 2. If m is *complete* (such as Lebesgue measure and LS measures) then, *for* $f\colon S \to E^*$ (E^n, C^n), m- and $\mathcal{M}$-measurability coincide (see Problem 3 below).

II. Measurability and Continuity. To study the connection between these notions, we first state two lemmas, often treated as *definitions*.

Lemma 1. *A map* $f\colon S \to E^n\,(C^n)$ *is* $\mathcal{M}$*-measurable on A iff*

$$A \cap f^{-1}[B] \in \mathcal{M}$$

for each Borel set (equivalently, open set) B in $E^n\,(C^n)$.

See Problems 8–10 in §2 for a sketch of the proof.

Lemma 2. *A map* $f\colon (S, \rho) \to (T, \rho')$ *is relatively continuous on* $A \subseteq S$ *iff for any open set* $B \subseteq (T, \rho')$, *the set* $A \cap f^{-1}[B]$ *is open in* (A, ρ) *as a subspace of* (S, ρ).

(This holds also with "*open*" replaced by "*closed.*")

Proof. By Chapter 4, §1, footnote 4, f is *relatively* continuous on A iff its restriction to A (call it $g\colon A \to T$) is continuous *in the ordinary sense.*

Now, by Problem 15(iv)(v) in Chapter 4, §2, with S replaced by A, this means that $g^{-1}[B]$ is open (closed) *in* (A, ρ) when B is so in (T, ρ'). But

$$g^{-1}[B] = \{x \in A \mid f(x) \in B\} = A \cap f^{-1}[B].$$

(Why?) Hence the result follows. $\square$

Theorem 1. *Let* $m\colon \mathcal{M} \to E^*$ *be a topological measure in* (S, ρ). *If* $f\colon S \to E^n\,(C^n)$ *is relatively continuous on a set* $A \in \mathcal{M}$, *it is* $\mathcal{M}$*-measurable on A.*

Proof. Let B be open in $E^n\,(C^n)$. By Lemma 2,

$$A \cap f^{-1}[B]$$

is open *in* (A, ρ). Hence by Theorem 4 of Chapter 3, §12,

$$A \cap f^{-1}[B] = A \cap U$$

for some open set U *in* (S, ρ).

Now, by assumption, A is in $\mathcal{M}$. So is U, as $\mathcal{M}$ is *topological* ($\mathcal{M} \supseteq \mathcal{G}$). Hence

$$A \cap f^{-1}[B] = A \cap U \in \mathcal{M}$$

for any open $B \subseteq E^n$ (C^n). The result follows by Lemma 1. $\square$

Note 3. The converse *fails*. For example, the Dirichlet function (Example (c) in Chapter 4, §1) is L-measurable (even simple) but discontinuous everywhere.

Note 4. Lemma 1 and Theorem 1 hold for a map $f\colon S \to (T, \rho')$, too, provided $f[A]$ is *separable*, i.e.,

$$f[A] \subseteq \overline{D}$$

for a countable set $D \subseteq T$ (cf. Problem 11 in §2).

***III.** For *strongly regular* measures (Definition 5 in Chapter 7, §7), we obtain the following theorem.

***Theorem 2** (Luzin)**.** *Let $m\colon \mathcal{M} \to E^*$ be a strongly regular measure in (S, ρ). Let $f\colon S \to (T, \rho')$ be m-measurable on A.*

Then given $\varepsilon > 0$, there is a closed set $F \subseteq A$ ($F \in \mathcal{M}$) such that

$$m(A - F) < \varepsilon$$

and f is relatively continuous on F.

(Note that if $T = E^*$, ρ' is as in Problem 5 of Chapter 3, §11.)

Proof.[1] By assumption, f is $\mathcal{M}$-measurable on a set

$$H = A - Q,\ mQ = 0;$$

so by Problem 7 in §1, $f[H]$ is *separable* in T. We may safely assume that f is $\mathcal{M}$-measurable *on* S and that *all* of T is separable. (If not, replace S and T by H and $f[H]$, restricting f to H, and m to $\mathcal{M}$-sets inside H; see also Problems 7 and 8 below.)

Then by Problem 12 of §2, we can fix globes $G_1, G_2, \ldots$ in T such that

(1) *each open set $B \neq \emptyset$ in T is the union of a subsequence of $\{G_k\}$.*

Now let $\varepsilon > 0$, and set

$$S_k = S \cap f^{-1}[G_k] = f^{-1}[G_k], \quad k = 1, 2, \ldots.$$

By Corollary 2 in §2, $S_k \in \mathcal{M}$. As m is strongly regular, we find for each S_k an open set

$$U_k \supseteq S_k,$$

---

[1] For a simpler proof, *in the case $mA < \infty$*, see Problem 10 below.

with $U_k \in \mathcal{M}$ and

$$m(U_k - S_k) < \frac{\varepsilon}{2^{k+1}}.$$

Let $B_k = U_k - S_k$, $D = \bigcup_k B_k$; so $D \in \mathcal{M}$ and

$$mD \leq \sum_k mB_k \leq \sum_k \frac{\varepsilon}{2^{k+1}} \leq \frac{1}{2}\varepsilon \tag{2}$$

and

$$U_k - B_k = S_k = f^{-1}[G_k]. \tag{2'}$$

As $D = \bigcup B_k$, we have

$$(\forall\, k) \quad B_k - D = B_k \cap (-D) = \emptyset.$$

Hence by (2′),

$$\begin{aligned}(\forall\, k) \quad f^{-1}[G_k] \cap (-D) &= (U_k - B_k) \cap (-D) \\ &= (U_k \cap (-D)) - (B_k \cap (-D)) = U_k \cap (-D).\end{aligned}$$

Combining this with (1), we have, for each open set $B = \bigcup_i G_{k_i}$ in T,

$$f^{-1}[B] \cap (-D) = \bigcup_i f^{-1}[G_{k_i}] \cap (-D) = \bigcup_i U_{k_i} \cap (-D). \tag{3}$$

Since the U_{k_i} are open in S (by construction), the set (3) is open in $S - D$ as a subspace of S. By Lemma 2, then, f is relatively continuous on $S - D$, or rather on

$$H - D = A - Q - D$$

(since we actually substituted S for H in the course of the proof). As $mQ = 0$ and $mD < \frac{1}{2}\varepsilon$ by (2),

$$m(H - D) < mA - \frac{1}{2}\varepsilon.$$

Finally, as m is strongly regular and $H - D \in \mathcal{M}$, there is a closed $\mathcal{M}$-set

$$F \subseteq H - D \subseteq A$$

such that

$$m(H - D - F) < \frac{1}{2}\varepsilon.$$

Since f is relatively continuous on $H - D$, it is surely so on F. Moreover,

$$A - F = (A - (H - D)) \cup (H - D - F);$$

so

$$m(A - F) \leq m(A - (H - D)) + m(H - D - F) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

This completes the proof. $\square$

***Lemma 3.** *Given $[a, b] \subset E^1$ and disjoint closed sets $A, B \subseteq (S, \rho)$, there always is a continuous map $g\colon S \to [a, b]$ such that $g = a$ on A and $g = b$ on B.*

Proof. If $A = \emptyset$ or $B = \emptyset$, set $g = b$ or $g = a$ on *all* of S.

If, however, A and B are both nonempty, set

$$g(x) = a + \frac{(b-a)\rho(x, A)}{\rho(x, A) + \rho(x, B)}.$$

As A is closed, $\rho(x, A) = 0$ *iff* $x \in A$ (Problem 15 in Chapter 3, §14); similarly for B. Thus $\rho(x, A) + \rho(x, B) \neq 0$.

Also, $g = a$ on A, $g = b$ on B, and $a \leq g \leq b$ on S.

For continuity, see Chapter 4, §8, Example (e). $\square$

***Lemma 4** (Tietze)**.** *If $f\colon (S, \rho) \to E^*$ is relatively continuous on a closed set $F \subseteq S$, there is a function $g\colon S \to E^*$ such that $g = f$ on F,*

$$\inf g[S] = \inf f[F], \ \sup g[S] = \sup f[F],$$

and g is continuous on all of S.

(We assume E^* metrized as in Problem 5 of Chapter 3, §11. If $|f| < \infty$, the *standard* metric in E^1 may be used.)

Proof Outline. First, assume $\inf f[F] = 0$ and $\sup f[F] = 1$. Set

$$A = F\left(f \leq \frac{1}{3}\right) = F \cap f^{-1}\left[\left[0, \frac{1}{3}\right]\right]$$

and

$$B = F\left(f \geq \frac{2}{3}\right) = F \cap f^{-1}\left[\left[\frac{2}{3}, 1\right]\right].$$

As F is closed *in* S, so are A and B by Lemma 2. (Why?)

As $B \cap A = \emptyset$, Lemma 3 yields a continuous map $g_1 : S \to [0, \frac{1}{3}]$, with $g_1 = 0$ on A, and $g_1 = \frac{1}{3}$ on B. Set $f_1 = f - g_1$ *on* F; so $|f_1| \leq \frac{2}{3}$, and f_1 is continuous on F.

Applying the same steps to f_1 (with suitable sets $A_1, B_1 \subseteq F$), find a continuous map g_2, with $0 \leq g_2 \leq \frac{2}{3} \cdot \frac{1}{3}$ on S. Then $f_2 = f_1 - g_2$ is continuous, and $0 \leq f_2 \leq (\frac{2}{3})^2$ on F.

Continuing, obtain two sequences $\{g_n\}$ and $\{f_n\}$ of real functions such that each g_n is continuous on S,

$$0 \leq g_n \leq \frac{1}{3}\left(\frac{2}{3}\right)^{n-1},$$

and $f_n = f_{n-1} - g_n$ is defined and continuous on F, with

$$0 \le f_n \le \left(\frac{2}{3}\right)^n$$

there ($f_0 = f$).

We claim that

$$g = \sum_{n=1}^{\infty} g_n$$

is the desired map.

Indeed, the series converges *uniformly* on S (Theorem 3 of Chapter 4, §12). As all g_n are continuous, so is g (Theorem 2 in Chapter 4, §12). Also,

$$\left| f - \sum_{k=1}^{n} g_k \right| \le \left(\frac{2}{3}\right)^n \to 0$$

on F (why?); so $f = g$ on F. Moreover,

$$0 \le g_1 \le g \le \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = 1 \text{ on } S.$$

Hence $\inf g[S] = 0$ and $\sup g[S] = 1$, as required.

Now assume

$$\inf f[F] = a < \sup f[F] = b \quad (a, b \in E^1).$$

Set

$$h(x) = \frac{f(x) - a}{b - a}$$

so that $\inf h[F] = 0$ and $\sup h[F] = 1$. (Why?)

As shown above, there is a continuous map g_0 on S, with

$$g_0 = h = \frac{f - a}{b - a}$$

on F, $\inf g_0[S] = 0$, and $\sup g_0[S] = 1$. Set

$$a + (b - a)g_0 = g.$$

Then g is the required function. (Verify!)

Finally, if $a, b \in E^*$ $(a < b)$, all reduces to the *bounded* case by considering $H(x) = \arctan f(x)$. $\square$

***Theorem 3** (Fréchet)**.** *Let $m\colon \mathcal{M} \to E^*$ be a strongly regular measure in (S, ρ). If $f\colon S \to E^*$ (E^n, C^n) is m-measurable on A, then*

$$f = \lim_{i \to \infty} f_i \ (a.e.(m)) \text{ on } A$$

for some sequence of maps f_i *continuous on* S. (*We assume* E^* *to be metrized as in Lemma* 4.)

Proof. We consider $f\colon S \to E^*$ (the other cases reduce to E^1 via components).

Taking $\varepsilon = \frac{1}{i}$ $(i = 1, 2, \ldots)$ in Theorem 2, we obtain for each i a closed $\mathcal{M}$-set $F_i \subseteq A$ such that

$$m(A - F_i) < \frac{1}{i}$$

and f is relatively continuous on each F_i. We may assume that $F_i \subseteq F_{i+1}$ (if not, replace F_i by $\bigcup_{k=1}^{i} F_k$).

Now, Lemma 4 yields for each i a continuous map $f_i\colon S \to E^*$ such that $f_i = f$ on F_i. We complete the proof by showing that $f_i \to f$ (*pointwise*) on the set

$$B = \bigcup_{i=1}^{\infty} F_i$$

and that $m(A - B) = 0$.

Indeed, fix any $x \in B$. Then $x \in F_i$ for some $i = i_0$, hence also for $i > i_0$ (since $\{F_i\}\!\uparrow$). As $f_i = f$ on F_i, we have

$$(\forall\, i > i_0) \quad f_i(x) = f(x),$$

and so $f_i(x) \to f(x)$ for $x \in B$. As $F_i \subseteq B$, we get

$$m(A - B) \le m(A - F_i) < \frac{1}{i}$$

for all i. Hence $m(A - B) = 0$, and all is proved. □

Problems on Measurable Functions in $(S, \mathcal{M}, m)$

1. Fill in all proof details in Corollaries 1 to 4.

1′. Verify Notes 3 and 4.

2. Prove Theorems 1 and 2 in §1 and Theorem 2 in §2, for *almost* measurable functions.

3. Prove Note 2.

[Hint: If $f\colon S \to E^*$ is $\mathcal{M}$-measurable on $B = A - Q$ $(mQ = 0,\ Q \subseteq A)$, then $A = B \cup Q$ and

$$(\forall\, a \in E^*) \quad A(f > a) = B(f > a) \cup Q(f > a).$$

Here $B(f > a) \in \mathcal{M}$ by Theorem 1 in §2, and $Q(f > a) \in \mathcal{M}$ if m is *complete*. For $f\colon S \to E^n$ (C^n), use Theorem 2 of §1.]

***4.** Show that if m is complete and $f\colon S \to (T, \rho')$ is m-measurable on A with $f[A]$ separable in T, then f is $\mathcal{M}$-measurable on A.

[Hint: Use Problem 13 in §2.]

***5.** Prove Theorem 1 for $f\colon S \to (T, \rho')$, assuming that $f[A]$ is separable in T.

6. Given $f_n \to f$ (a.e.) on A, prove that $f_n \to g$ (a.e.) on A iff $f = g$ (a.e.) on A.

7. Given $A \in \mathcal{M}$ in $(S, \mathcal{M}, m)$, let m_A be the restriction of m to

$$\mathcal{M}_A = \{X \in \mathcal{M} \mid X \subseteq A\}.$$

Prove that

(i) $(A, \mathcal{M}_A, m_A)$ is a measure space (called a *subspace* of $(S, \mathcal{M}, m)$);

(ii) if m is complete, topological, σ-finite or (strongly) regular, so is m_A.

8. (i) Show that if $D \subseteq K \subseteq (T, \rho')$, then the closure of D *in the subspace* (K, ρ') is $K \cap \overline{D}$, where $\overline{D}$ is the closure of D *in* (T, ρ').
[Hint: Use Problem 11 in Chapter 3, §16.]

(ii) Prove that if $B \subseteq K$ and if B is separable in (T, ρ'), it is so in (K, ρ').
[Hint: Use Problem 7 from §1.]

***9.** Fill in all proof details in Lemma 4.

10. Simplify the proof of Theorem 2 for the case $mA < \infty$.
[Outline: (i) First, let f be *elementary*, with $f = a_i$ on $A_i \in \mathcal{M}$, $A = \bigcup_i A_i$ (*disjoint*), $\sum mA_i = mA < \infty$.

Given $\varepsilon > 0$,

$$(\exists\, n) \quad mA - \sum_{i=1}^{n} mA_i < \frac{1}{2}\varepsilon.$$

Each A_i has a closed subset $F_i \in \mathcal{M}$ with $m(A_i - F_i) < \varepsilon/2n$. (Why?) Now use Problem 17 in Chapter 4, §8, and set $F = \bigcup_{i=1}^{n} F_i$.

(ii) If f is $\mathcal{M}$-*measurable* on $H = A - Q$, $mQ = 0$, then by Theorem 3 in §1, $f_n \to f$ (*uniformly*) on H for some elementary maps f_n. By (i), each f_n is relatively continuous on a closed $\mathcal{M}$-set $F_n \subseteq H$, with $mH - mF_n < \varepsilon/2^n$; so *all* f_n are relatively continuous on $F = \bigcap_{n=1}^{\infty} F_n$. Show that F is the required set.]

11. Given $f_n\colon S \to (T, \rho')$, $n = 1, 2, \ldots,$ we say that

(i) $f_n \to f$ *almost uniformly* on $A \subseteq S$ iff

$$(\forall\, \delta > 0)\ (\exists\, D \in \mathcal{M} \mid mD < \delta) \quad f_n \to f \text{ (uniformly) on } A - D;$$

(ii) $f_n \to f$ *in measure* on A iff

$$(\forall\, \delta, \sigma > 0)\ (\exists\, k)\ (\forall\, n > k)\ (\exists\, D_n \in \mathcal{M} \mid mD_n < \delta) \quad \rho'(f, f_n) < \sigma \text{ on } A - D_n.$$

Prove the following.

(a) $f_n \to f$ (*uniformly*) implies $f_n \to f$ (*almost uniformly*), and the latter implies both $f_n \to f$ (*in measure*) and $f_n \to f$ (*a.e.*).

(b) Given $f_n \to f$ (*almost uniformly*), we have $f_n \to g$ (*almost uniformly*) iff $f = g$ (a.e.); similarly for convergence in measure.

(c) If f and f_n are $\mathcal{M}$-measurable on A, then $f_n \to f$ in measure on A iff

$$(\forall \sigma > 0) \quad \lim_{n\to\infty} mA(\rho'(f, f_n) \geq \sigma) = 0.$$

12. Assuming that $f_n \colon S \to (T, \rho')$ is m-measurable on A for $n = 1, 2, \ldots,$ that $mA < \infty$, and that $f_n \to f$ (*a.e.*) on A, prove the following.

(i) Lebesgue's theorem: $f_n \to f$ (*in measure*) on A (see Problem 11).

(ii) Egorov's theorem: $f_n \to f$ (*almost uniformly*) on A.

[Outline: (i) f_n and f are $\mathcal{M}$-measurable on $H = A - Q$, $mQ = 0$ (Corollary 1), with $f_n \to f$ (*pointwise*) on H. For all i, k, set

$$H_i(k) = \bigcap_{n=i}^{\infty} H\left(\rho'(f_n, f) < \frac{1}{k}\right) \in \mathcal{M}$$

by Problem 6 in §1. Show that $(\forall k)\ H_i(k) \nearrow H$; hence

$$\lim_{i\to\infty} mH_i(k) = mH = mA < \infty;$$

so

$$(\forall \delta > 0)\ (\forall k)\ (\exists i_k) \quad m(A - H_{i_k}(k)) < \frac{\delta}{2^k},$$

proving (i), since

$$(\forall n > i_k) \quad \rho'(f_n, f) < \frac{1}{k} \text{ on } H_{i_k}(k) = A - (A - H_{i_k}(k)).$$

(ii) Continuing, set $(\forall k)\ D_k = H_{i_k}(k)$ and

$$D = A - \bigcap_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty} (A - D_k).$$

Deduce that $D \in \mathcal{M}$ and

$$mD \leq \sum_{k=1}^{\infty} m(A - H_{i_k}(k)) < \sum_{k=1}^{\infty} \frac{\delta}{2^k} = \delta.$$

Now, from the definition of the $H_i(k)$, show that $f_n \to f$ (*uniformly*) on $A - D$, proving (ii).]

13. Disprove the *converse* to Problem 12(i).

[Outline: Assume that $A = [0, 1)$; for all $0 \leq k$ and all $0 \leq i < 2^k$, set

$$g_{ik}(x) = \begin{cases} 1 & \text{if } \frac{i-1}{2^k} \leq x < \frac{i}{2^k}, \\ 0 & \text{otherwise.} \end{cases}$$

Put the g_{ik} in a *single* sequence by

$$f_{2^k+i} = g_{ik}.$$

Show that $f_n \to 0$ in L measure on A, yet for *no* $x \in A$ does $f_n(x)$ converge as $n \to \infty$.]

14. Prove that if $f\colon S \to (T, \rho')$ is m-measurable on A and $g\colon T \to (U, \rho'')$ is relatively continuous on $f[A]$, then $g \circ f\colon S \to (U, \rho'')$ is m-measurable on A.

[Hint: Use Corollary 4 in §1.]

§4. Integration of Elementary Functions

In Chapter 5, integration was treated as antidifferentiation. Now we adopt another, *measure-theoretical* approach.

Lebesgue's original theory was based on *Lebesgue measure* (Chapter 7, §8). The more general modern treatment develops the integral for functions $f\colon S \to E$ in an *arbitrary* measure space. Henceforth, $(S, \mathcal{M}, m)$ is fixed, and the range space E is E^1, E^*, C, E^n, or another *complete* normed space. Recall that in such a space, $\sum_i |a_i| < \infty$ implies that $\sum a_i$ *converges* and is *permutable* (Chapter 7, §2).

We start with *elementary* maps, including *simple* maps as a special case.[1]

Definition 1.

Let $f\colon S \to E$ be elementary on $A \in \mathcal{M}$; so $f = a_i$ on A_i for some $\mathcal{M}$-partition

$$A = \bigcup_i A_i \ (\textit{disjoint}).$$

(Note that there may be *many* such partitions.)

We say that f is *integrable* (with respect to m), or *m-integrable*, on A iff

$$\sum |a_i|\, mA_i < \infty.$$

(The notation "$|a_i|\, mA_i$" always makes sense by our conventions (2^*) in Chapter 4, §4.) If m is Lebesgue measure, then we say that f is Lebesgue integrable, or L-integrable.

We then define $\int_A f$, the *m-integral* of f on A, by

$$\int_A f = \int_A f\, dm = \sum_i a_i\, mA_i. \tag{1}$$

---

[1] For a "limited approach," use *finite* $\mathcal{M}$-partitions and $\mathcal{M}$-*simple* maps, treating m as an additive premeasure on $\mathcal{M}$, a *ring*.

(The notation "dm" is used to specify the measure m.)

The "classical" notation for $\int_A f\,dm$ is $\int_A f(x)\,dm(x)$.

Note 1. The assumption

$$\sum |a_i|\,mA_i < \infty$$

implies

$$(\forall\, i) \quad |a_i|\,mA_i < \infty;$$

so $a_i = 0$ if $mA_i = \infty$, and $mA_i = 0$ if $|a_i| = \infty$. Thus by our conventions, all "bad" terms $a_i\,mA_i$ *vanish*. Hence the sum in (1) makes sense and is *finite*.

Note 2. This sum is also *independent of the particular choice of* $\{A_i\}$. For if $\{B_k\}$ is another $\mathcal{M}$-partition of A, with $f = b_k$ on B_k, say, then $f = a_i = b_k$ on $A_i \cap B_k$ whenever $A_i \cap B_k \neq \emptyset$. Also,

$$(\forall\, i) \quad A_i = \bigcup_k (A_i \cap B_k) \text{ } (\textit{disjoint});$$

so

$$(\forall\, i) \quad a_i\,mA_i = \sum_k a_i\,m(A_i \cap B_k),$$

and hence (see Theorem 2 of Chapter 7, §2, and Problem 11 there)

$$\sum_i a_i\,mA_i = \sum_i \sum_k a_i\,m(A_i \cap B_k) = \sum_k \sum_i b_k\,m(A_i \cap B_k) = \sum_k b_k\,mB_k.$$

(Explain!)

This makes our definition (1) *unambiguous* and allows us to choose *any* $\mathcal{M}$-partition $\{A_i\}$, with f constant on each A_i, when forming integrals (1).

Corollary 1. *Let $f\colon S \to E$ be elementary and integrable on $A \in \mathcal{M}$. Then the following statements are true.*

(i) $|f| < \infty$ *a.e. on* A.[2]

(ii) f *and* $|f|$ *are elementary and integrable on any* $\mathcal{M}$*-set* $B \subseteq A$, *and*

$$\left| \int_B f \right| \le \int_B |f| \le \int_A |f|.$$

(iii) *The set* $B = A(f \neq 0)$ *is* σ*-finite* (*Definition* 4 *in Chapter* 7, §5), *and*

$$\int_A f = \int_B f.$$

---

[2] That is, on $A - Q$ for some $Q \in \mathcal{M}$, with $mQ = 0$.

(iv) *If* $f = a$ *(constant) on* A,

$$\int_A f = a \cdot mA.$$

(v) $\int_A |f| = 0$ *iff* $f = 0$ *a.e. on* A.

(vi) *If* $mQ = 0$, *then*

$$\int_A f = \int_{A-Q} f$$

(so we may neglect sets of measure 0 *in integrals).*

(vii) *For any* k *in the scalar field of* E, kf *is elementary and integrable, and*

$$\int_A kf = k \int_A f.$$

Note that if f is scalar valued, k may be a vector. If $E = E^*$, we assume $k \in E^1$.

Proof.

(i) By Note 1, $|f| = |a_i| = \infty$ only on those A_i with $mA_i = 0$. Let Q be the union of all such A_i. Then $mQ = 0$ and $|f| < \infty$ on $A - Q$, proving (i).

(ii) If $\{A_i\}$ is an $\mathcal{M}$-partition of A, $\{B \cap A_i\}$ is one for B. (Verify!) We have $f = a_i$ and $|f| = |a_i|$ on $B \cap A_i \subseteq A_i$.

Also,

$$\sum |a_i| \, m(B \cap A_i) \leq \sum |a_i| \, mA_i < \infty.$$

(Why?) Thus f and $|f|$ are elementary and integrable *on* B, and (ii) easily follows by formula (1).

(iii) By Note 1, $f = 0$ on A_i if $mA_i = \infty$. Thus $f \neq 0$ on A_i only if $mA_i < \infty$. Let $\{A_{i_k}\}$ be the subsequence of those A_i on which $f \neq 0$; so

$$(\forall\, k) \quad mA_{i_k} < \infty.$$

Also,

$$B = A(f \neq 0) = \bigcup_k A_{i_k} \in \mathcal{M} \text{ (σ-finite!).}$$

By (ii), f is elementary and integrable on B. Also,

$$\int_B f = \sum_k a_{i_k} \, mA_{i_k},$$

while

$$\int_A f = \sum_i a_i \, mA_i.$$

These sums differ only by terms with $a_i = 0$. Thus (iii) *follows.*

The proof of (iv)–(vii) is left to the reader. □

Note 3. If $f\colon S \to E^*$ is elementary and *sign-constant* on A, we also allow that

$$\int_A f = \sum_i a_i \, mA_i = \pm\infty.$$

Thus here $\int_A f$ exists even if f is not integrable. Apart from claims of integrability and σ-finiteness, Corollary 1(ii)–(vii) hold for such f, with the same proofs.

Example.

Let m be Lebesgue measure in E^1. Define $f = 1$ on R (rationals) and $f = 0$ on $E^1 - R$; see Chapter 4, §1, Example (c). Let $A = [0,1]$.

By Corollary 1 in Chapter 7, §8, $A \cap R \in \mathcal{M}^*$ and $m(A \cap R) = 0$. Also, $A - R \in \mathcal{M}^*$.

Thus $\{A \cap R, A - R\}$ is an $\mathcal{M}^*$-partition of A, with $f = 1$ on $A \cap R$ and $f = 0$ on $A - R$.

Hence f is elementary and integrable on A, and

$$\int_A f = 1 \cdot m(A \cap R) + 0 \cdot m(A - R) = 0.$$

Thus f is L-integrable (even though it is *nowhere continuous*).

Theorem 1 (additivity)**.**

(i) *If $f\colon S \to E$ is elementary and integrable or elementary and nonnegative on $A \in \mathcal{M}$, then*

$$\int_A f = \sum_k \int_{B_k} f \tag{2}$$

for any $\mathcal{M}$-partition $\{B_k\}$ of A.

(ii) *If f is elementary and integrable on each set B_k of a* finite *$\mathcal{M}$-partition*

$$A = \bigcup_k B_k,$$

it is elementary and integrable on all of A, and (2) *holds again.*

Proof. (i) If f is elementary and integrable or elementary and nonnegative on $A = \bigcup_k B_k$, it is surely so on each B_k by Corollary 2 of §1 and Corollary 1(ii) above.

Thus for each k, we can fix an $\mathcal{M}$-partition $B_k = \bigcup_i A_{ki}$, with f constant ($f = a_{ki}$) on A_{ki}, $i = 1, 2, \ldots$. Then

$$A = \bigcup_k B_k = \bigcup_k \bigcup_i A_{ki}$$

is an $\mathcal{M}$-partition *of* A into the disjoint sets $A_{ki} \in \mathcal{M}$.

Now, by definition,

$$\int_{B_k} f = \sum_i a_{ki}\, mA_{ki}$$

and

$$\int_A f = \sum_{k,i} a_{ki}\, mA_{ki} = \sum_k \Bigl(\sum_i a_{ki}\, mA_{ki}\Bigr) = \sum_k \int_{B_k} f$$

by rules for double series. This proves formula (2).

(ii) If f is elementary and integrable on B_k ($k = 1, \ldots, n$), then with the same notation, we have

$$\sum_i |a_{ki}|\, mA_{ki} < \infty$$

(by integrability); hence

$$\sum_{k=1}^{n} \sum_i |a_{ki}|\, mA_{ki} < \infty.$$

This means, however, that f is elementary and integrable *on* A, and so clause (ii) follows. □

Caution. Clause (ii) fails if the partition $\{B_k\}$ is *infinite*.

Theorem 2.

(i) *If* $f, g\colon S \to E^*$ *are elementary and nonnegative on* A, *then*

$$\int_A (f+g) = \int_A f + \int_A g.$$

(ii) *If* $f, g\colon S \to E$ *are elementary and integrable on* A, *so is* $f \pm g$, *and*

$$\int_A (f \pm g) = \int_A f \pm \int_A g.$$

Proof. Arguing as in the proof of Theorem 1 of §1, we can make f and g constant on sets of *one and the same* $\mathcal{M}$-partition of A, say, $f = a_i$ and $g = b_i$ on $A_i \in \mathcal{M}$; so

$$f \pm g = a_i \pm b_i \text{ on } A_i, \quad i = 1, 2, \ldots.$$

In case (i), $f, g \geq 0$; so integrability is irrelevant by Note 3, and formula (1) yields

$$\int_A (f+g) = \sum_i (a_i + b_i)\, mA_i = \sum_i a_i\, mA_i + \sum b_i\, mA_i = \int_A f + \int_A g.$$

In (ii), we similarly obtain

$$\sum_i |a_i \pm b_i|\, mA_i \leq \sum |a_i|\, mA_i + \sum_i |b_i|\, mA_i < \infty.$$

(Why?) Thus $f \pm g$ is elementary and integrable on A. As before, we also get

$$\int_A (f \pm g) = \int_A f \pm \int_A g,$$

simply by rules for addition of convergent series. (Verify!) $\square$

Note 4. As we know, the *characteristic function* C_B of a set $B \subseteq S$ is defined

$$C_B(x) = \begin{cases} 1, & x \in B, \\ 0, & x \in S - B. \end{cases}$$

If $g\colon S \to E$ is elementary on A, so that

$$g = a_i \text{ on } A_i,\ 1, 2, \ldots,$$

for some $\mathcal{M}$-partition

$$A = \bigcup A_i,$$

then

$$g = \sum_i a_i C_{A_i} \text{ on } A.$$

(This sum always exists for *disjoint* sets A_i. Why?) We shall often use this notation.

If m is Lebesgue measure in E^1, the integral

$$\int_A g = \sum_i a_i\, mA_i$$

has a simple geometric interpretation; see Figure 33. Let $A = [a, b] \subset E^1$; let g be bounded and nonnegative on E^1. Each product $a_i\, mA_i$ is the area of a rectangle with base A_i and altitude a_i. (We assume the A_i to be

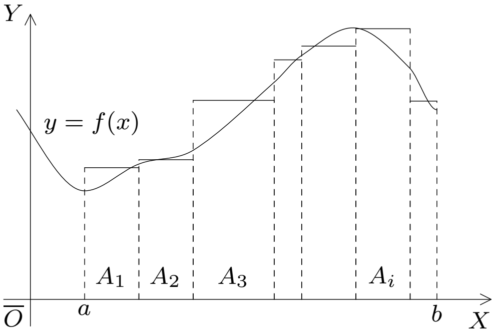

FIGURE 33

intervals here.) The total area,

$$\int_A g = \sum_i a_i \, mA_i,$$

can be treated as an approximation to the *area under some curve* $y = f(x)$, where f is approximated by g (Theorem 3 in §1). Integration historically arose from such approximations.

Integration of elementary extended-real functions. Note 3 can be extended to *sign-changing* functions as follows.

Definition 2.

If

$$f = \sum_i a_i \, C_{A_i} \quad (a_i \in E^*)$$

on

$$A = \bigcup_i A_i \quad (A_i \in \mathcal{M}),$$

we set

$$\int_A f = \int_A f^+ - \int_A f^-, \tag{3}$$

with

$$f^+ = f \vee 0 \geq 0 \text{ and } f^- = (-f) \vee 0 \geq 0;$$

see §2.

By Theorem 2 in §2, f^+ and f^- are *elementary and nonnegative* on A; so

$$\int_A f^+ \text{ and } \int_A f^-$$

are *defined* by Note 3, and *so is*

$$\int_A f = \int_A f^+ - \int_A f^-$$

by our conventions (2*) *in Chapter* 4, §4.

We shall have use for formula (3), even if

$$\int_A f^+ = \int_A f^- = \infty;$$

then we say that $\int_A f$ is *unorthodox* and equate it to $+\infty$, *by convention*; cf. Chapter 4, §4. (Other integrals are called *orthodox*.) Thus for *elementary and* (*extended*) *real* functions, $\int_A f$ is *always* defined. (We further develop this idea in §5.)

Note 5. With f as above, we clearly have

$$f^+ = a_i^+ \text{ and } f^- = a_i^- \text{ on } A_i,$$

where

$$a_i^+ = \max(a_i, 0) \text{ and } a_i^- = \max(-a_i, 0).$$

Thus

$$\int_A f^+ = \sum a_i^+ \cdot mA_i \text{ and } \int_A f^- = \sum a_i^- \cdot mA_i,$$

so that

$$\int_A f = \int_A f^+ - \int_A f^- = \sum_i a_i^+ \cdot mA_i - \sum_i a_i^- \cdot mA_i. \tag{4}$$

If $\int_A f^+ < \infty$ or $\int_A f^- < \infty$, we can subtract the two series *termwise* (Problem 14 of Chapter 4, §13) to obtain

$$\int_A f = \sum_i (a_i^+ - a_i^-)\, mA_i = \sum_i a_i\, mA_i$$

for $a_i^+ - a_i^- = a_i$. Thus formulas (3) and (4) *agree with our previous definitions.*[3]

Problems on Integration of Elementary Functions

1. Verify Note 2.

1′. Prove Corollary 1(iv)–(vii).

2. Prove that $\int_A f = 0$ *if* $mA = 0$ or $f = 0$ on A. Disprove the converse by examples.

3. Find a *primitive* F for $f = C_R$ in our example. Show that

$$\int_{[0,1]} f\, dm = F(1) - F(0).$$

4. Fill in the proof details in Theorem 2.
[Hint: Use comparison test for series.]

⇒**5.** Show that if f and g are elementary and nonnegative with $f \geq g$ on A, then

$$\int_A f \geq \int_A g \geq 0.$$

[Hint: As in Theorem 2, let

$$f = \sum_i a_i\, C_{A_i} \text{ and } g = \sum_i b_i\, C_{A_i}.$$

Then $f \geq g \geq 0$ implies $a_i \geq b_i \geq 0$.]

---

[3] For a "limited approach," pass from here to §9.

⇒**6.** Prove that if f and g are elementary and (extended) real on A, then

$$\int_A (f \pm g) = \int_A f \pm \int_A g,$$

provided

(i) $\int_A f$ or $\int_A g$ is finite, or

(ii) $\int_A f$, $\int_A g$, and $\int_A f \pm \int_A g$ are all *orthodox*.

[Outline: As in Theorem 2, let

$$f = \sum_i a_i C_{A_i} \text{ and } g = \sum_i b_i C_{A_i},$$

so

$$f \pm g = a_i \pm b_i \text{ on } A_i.$$

Now, if

$$\left|\int_A f\right| < \infty,$$

then by Problem 14 in Chapter 4, §13, and formula (4), $\sum a_i \, mA_i$ converges *absolutely*; so its termwise addition to any other series does not affect the absolute convergence or divergence of the latter, i.e., the finiteness or infiniteness of its positive and negative parts. For example,

$$\sum_i (a_i \pm b_i)^+ \, mA_i = \infty$$

iff

$$\sum b_i^+ \, mA_i = \infty.$$

Thus if

$$\int_A g = \pm\infty,$$

then

$$\int_A (f \pm g) = \int_A g = \pm\infty = \int_A f \pm \int_A g.$$

If *both*

$$\int_A f, \int_A g \neq \pm\infty,$$

Theorem 2(ii) applies. In the orthodox infinite case, a similar proof works on noting that either the positive or the negative parts of *both* series are finite if

$$\int_A f \pm \int_A g$$

is orthodox, too. (Verify!)]

7. Show that if f is elementary and nonnegative on A and

$$\int_A f > p \in E^*,$$

then there is an elementary and nonnegative map g on A such that

$$\int_A f \geq \int_A g > p,$$

$g = 0$ on $A(f = 0)$, and

$$f > g \text{ on } A - A(f = 0).$$

[Hints: Let

$$B = A(f = \infty)$$

and

$$C = A - B;$$

so $B, C \in \mathcal{M}$ (Corollary 2 in §2). For all $n > 0$, define

$$g_n = n \text{ on } B$$

and

$$g_n = \left(1 - \frac{1}{n}\right) f \text{ on } C;$$

so g_n is elementary and nonnegative on A and

$$f > g_n \text{ on } A - A(f = 0). \text{ (Why?)}$$

By Theorem 1 and Corollary 1(iv)(vii),

$$\int_A g_n = \int_B g_n + \int_C g_n = \int_B (n) + \int_C \left(1 - \frac{1}{n}\right) f = n \cdot mB + \left(1 - \frac{1}{n}\right) \int_C f.$$

Deduce that

$$\lim_{n \to \infty} \int_A g_n = \int_B f + \int_C f = \int_A f > p;$$

so

$$(\exists\, n) \quad \int_A g_n > p.$$

Take $g = g_n$ for *that* n.]

8. Show that if $E = E^*$, Theorem 1(i) holds also if $\int_A f$ is infinite but *orthodox*.

9. (i) Prove that if f is *elementary and integrable* on A, so is $-f$, and

$$\int_A (-f) = -\int_A f.$$

(ii) Show that this holds also if f is *elementary and (extended) real* and $\int_A f$ is *orthodox*.

§5. Integration of Extended-Real Functions

We shall now define integrals for *arbitrary* functions $f\colon S \to E^*$ in a measure space $(S, \mathcal{M}, m)$.[1] We start with the case $f \geq 0$.

Definition 1.

Given $f \geq 0$ on $A \in \mathcal{M}$, we define the *upper* and *lower integrals*,

$$\overline{\int} \text{ and } \underline{\int},$$

of f on A (with respect to m) by

$$\overline{\int}_A f = \overline{\int}_A f\,dm = \inf_h \int_A h \tag{1'}$$

over all elementary maps $h \geq f$ on A, and

$$\underline{\int}_A f = \underline{\int}_A f\,dm = \sup_g \int_A g \tag{1''}$$

over all elementary and nonnegative maps $g \leq f$ on A.

If f is not nonnegative, we use $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$ (§2), and set

$$\begin{aligned} \overline{\int}_A f &= \overline{\int}_A f\,dm = \overline{\int}_A f^+ - \underline{\int}_A f^- \text{ and} \\ \underline{\int}_A f &= \underline{\int}_A f\,dm = \underline{\int}_A f^+ - \overline{\int}_A f^-. \end{aligned} \tag{1}$$

By our conventions, these expressions are *always* defined. The integral $\overline{\int}_A f$ (or $\underline{\int}_A f$) is called *orthodox* iff it does not have the form $\infty - \infty$ in (1), e.g., if $f \geq 0$ (i.e., $f^- = 0$), or if $\int_A f < \infty$. *An unorthodox integral equals* $+\infty$.

We often write $\int$ for $\overline{\int}$ and call it simply *the* integral (of f), even if

$$\overline{\int}_A f \neq \underline{\int}_A f.^2$$

"Classical" notation is $\int_A f(x)\,dm(x)$.

---

[1] Those who wish to consider *measurable* maps only should take Theorem 3 *earlier*.

[2] There is good reason for identifying "*integral*" with "*upper integral*."

Definition 2.

The function f is called *integrable* (or m*-integrable*, or *Lebesgue integrable, with respect to* m) on A, iff

$$\overline{\int}_A f\,dm = \underline{\int}_A f\,dm \neq \pm\infty.$$

The process described above is called (abstract) *Lebesgue integration* as opposed to *Riemann integration* (B. Riemann, 1826–1866). The latter deals with bounded functions only and allows h and g in $(1')$ and $(1'')$ to be *simple step functions* only (see §9). It is inferior to Lebesgue theory.

The values of

$$\overline{\int}_A f\,dm \text{ and } \underline{\int}_A f\,dm$$

depend on m. If m is Lebesgue measure, we speak of *Lebesgue integrals, in the stricter sense.* If m is Lebesgue–Stieltjes measure, we speak of *LS-integrals*, and so on.

Note 1. If f is *elementary and* (*extended*) *real*, our present definition of

$$\overline{\int}_A f$$

agrees with that of §4. For if $f \geq 0$, f itself is the least of all elementary and nonnegative functions

$$h \geq f$$

and the greatest of all elementary and nonnegative functions

$$g \leq f.$$

Thus by Problem 5 in §4,

$$\int_A f = \min_{h \geq f} \int_A h = \max_{g \leq f} \int_A g,$$

i.e.,

$$\int_A f = \overline{\int}_A f = \underline{\int}_A f.$$

If, however, $f \ngeq 0$, this follows by Definition 2 in §4. This also shows that *for elementary and* (*extended*) *real maps,*

$$\overline{\int}_A f = \underline{\int}_A f \textit{ always.}$$

(See also Theorem 3.)

Note 2. By Definition 1,

$$\underline{\int}_A f \le \overline{\int}_A f \text{ always.}$$

For if $f \ge 0$, then for any elementary and nonnegative maps g, h with

$$g \le f \le h,$$

we have

$$\int_A g \le \int_A h$$

by Problem 5 in §4. Thus

$$\underline{\int}_A f = \sup_g \int_A g$$

is a *lower bound* of all such $\int_A h$, and so

$$\underline{\int}_A f \le \operatorname{glb} \int_A h = \overline{\int}_A f.$$

In the general formula (1), too,

$$\underline{\int}_A f \le \overline{\int}_A f,$$

since

$$\underline{\int}_A f^+ \le \overline{\int}_A f^+ \text{ and } \underline{\int}_A f^- \le \overline{\int}_A f^-.$$

Theorem 1. *For any functions $f, g \colon S \to E^*$ and any set $A \in \mathcal{M}$, we have the following results.*[3]

(a) *If $f = a$ (constant) on A, then*

$$\overline{\int}_A f = \underline{\int}_A f = a \cdot mA.$$

(b) *If $f = 0$ on A or $mA = 0$, then*

$$\overline{\int}_A f = \underline{\int}_A f = 0.$$

(c) *If $f \ge g$ on A, then*

$$\overline{\int}_A f \ge \overline{\int}_A g \text{ and } \underline{\int}_A f \ge \underline{\int}_A g.$$

---

[3] Note that integrability is redundant here and in Theorem 2.

(d) *If $f \geq 0$ on A, then*

$$\overline{\int}_A f \geq 0 \text{ and } \underline{\int}_A f \geq 0.$$

Similarly if $f \leq 0$ on A.

(e) *If $0 \leq p < \infty$, then*

$$\overline{\int}_A pf = p\overline{\int}_A f \text{ and } \underline{\int}_A pf = p\underline{\int}_A f.$$

(e′) *We have*

$$\overline{\int}_A (-f) = -\underline{\int}_A f \text{ and } \underline{\int}_A (-f) = -\overline{\int}_A f$$

if one of the integrals involved in each case is orthodox. Otherwise,

$$\overline{\int}_A (-f) = \infty = \underline{\int}_A f \text{ and } \underline{\int}_A (-f) = \infty = \overline{\int}_A f.$$

(f) *If $f \geq 0$ on A and*

$$A \supseteq B,\ B \in \mathcal{M},$$

then

$$\overline{\int}_A f \geq \overline{\int}_B f \text{ and } \underline{\int}_A f \geq \underline{\int}_B f.$$

(g) *We have*

$$\left|\overline{\int}_A f\right| \leq \overline{\int}_A |f| \text{ and } \left|\underline{\int}_A f\right| \leq \overline{\int}_A |f|$$

(but not

$$\left|\underline{\int}_A f\right| \leq \underline{\int}_A |f|$$

in general).

(h) *If $f \geq 0$ on A and $\overline{\int}_A f = 0$ (or $f \leq 0$ and $\underline{\int}_A f = 0$), then $f = 0$ a.e. on A.*

Proof. We prove only some of the above, leaving the rest to the reader.

(a) This following by Corollary 1(iv) in §4.

(b) Use (a) and Corollary 1(v) in §4.

(c) First, let

$$f \geq g \geq 0 \text{ on } A.$$

Take any elementary and nonnegative map $H \geq f$ on A. Then $H \geq g$ as well; so by definition,

$$\overline{\int}_A g = \inf_{h \geq g} \int_A h \leq \int_A H.$$

Thus

$$\overline{\int}_A f \leq \int_A H$$

for *any* such H. Hence also

$$\overline{\int}_A g \leq \inf_{H \geq f} \int_A H = \overline{\int}_A f.$$

Similarly,

$$\underline{\int}_A f \geq \underline{\int}_A g$$

if $f \geq g \geq 0$.

In the general case, $f \geq g$ implies

$$f^+ \geq g^+ \text{ and } f^- \leq g^-. \text{ (Why?)}$$

Thus by what was proved above,

$$\overline{\int}_A f^+ \geq \overline{\int}_A g^+ \text{ and } \underline{\int}_A f^- \leq \underline{\int}_A g^-.$$

Hence

$$\overline{\int}_A f^+ - \underline{\int}_A f^- \geq \overline{\int}_A g^+ - \underline{\int}_A g^-;$$

i.e.,

$$\overline{\int}_A f \geq \overline{\int}_A g.$$

Similarly, one obtains

$$\underline{\int}_A f \geq \underline{\int}_A g.$$

(d) It is clear that (c) implies (d).

(e) Let $0 \leq p < \infty$ and suppose $f \geq 0$ on A. Take any elementary and nonnegative map

$$h \geq f \text{ on } A.$$

By Corollary 1(vii) and Note 3 of §4,

$$\int_A ph = p\int_A h$$

for *any* such h. Hence

$$\overline{\int}_A pf = \inf_h \int_A ph = \inf_h p\int_A h = p\overline{\int}_A f.$$

Similarly,

$$\underline{\int}_A pf = p\underline{\int}_A f.$$

The general case reduces to the case $f \geq 0$ by formula (1).

(e′) Assertion (e′) follows from (1) since

$$(-f)^+ = f^-, \quad (-f)^- = f^+,$$

and $-(x-y) = y-x$ if $x-y$ is *orthodox*. (Why?)

(f) Take any elementary and nonnegative map

$$h \geq f \geq 0 \text{ on } A.$$

By Corollary 1(ii) and Note 3 of §4,

$$\int_B h \geq \int_A h$$

for *any* such h. Hence

$$\overline{\int}_B f = \inf_h \int_B h \leq \inf_h \int_A h = \overline{\int}_A f.$$

Similarly for $\underline{\int}$.

(g) This follows from (c) and (e′) since $\pm f \leq |f|$ implies

$$\overline{\int}_A |f| \geq \overline{\int}_A f \geq \underline{\int}_A f$$

and

$$\overline{\int}_A |f| \geq \overline{\int}_A (-f) \geq -\underline{\int}_A f \geq -\overline{\int}_A f. \quad \square$$

For (h) and later work, we need the following lemmas.

Lemma 1. *Let* $f\colon S \to E^*$ *and* $A \in \mathcal{M}$. *Then the following are true.*

(i) *If*

$$\int_A f < q \in E^*,$$

there is an elementary and (*extended*) *real map*

$$h \geq f \text{ on } A,$$

with

$$\int_A h < q.$$

(ii) *If*

$$\int_A f > p \in E^*,$$

there is an elementary and (*extended*) *real map*

$$g \leq f \text{ on } A,$$

with

$$\int_A g > p;$$

moreover, g *can be made elementary and nonnegative if* $f \geq 0$ *on* A.

Proof. If $f \geq 0$, this is immediate by Definition 1 and the properties of glb and lub.

If, however, $f \ngeq 0$, and if

$$q > \int_A f = \overline{\int}_A f^+ - \underline{\int}_A f^-,$$

our conventions yield

$$\infty > \int_A f^+. \text{ (Why?)}$$

Thus there are $u, v \in E^*$ such that $q = u + v$ and

$$0 \leq \int_A f^+ < u < \infty$$

and

$$-\int_A f^- < v.$$

To see why this is so, choose u so close to $\overline{\int}_A f^+$ that

$$q - u > -\underline{\int}_A f^-$$

and set $v = q - u$.

As the lemma holds for *positive* functions, we find elementary and nonnegative maps h' and h'', with

$$h' \geq f^+,\ h'' \leq f^-,$$

$$\int_A h' < u < \infty \text{ and } \int_A h'' > -v.$$

Let $h = h' - h''$. Then

$$h \geq f^+ - f^- = f,$$

and by Problem 6 in §4,

$$\int_A h = \int_A h' - \int_A h'' \quad \left(\text{for } \int_A h' \text{ is } \textit{finite}!\right).$$

Hence

$$\int_A h > u + v = q,$$

and clause (i) is proved in full.

Clause (ii) follows from (i) by Theorem 1(e′) if

$$\underline{\int}_A f < \infty.$$

(Verify!) For the case $\underline{\int}_A f = \infty$, see Problem 3. □

Note 3. The preceding lemma shows that formulas (1′) and (1″) hold (and might be used as *definitions*) even for *sign-changing* f, g, and h.

Lemma 2. *If $f\colon S \to E^*$ and $A \in \mathcal{M}$, there are $\mathcal{M}$-measurable maps g and h, with*

$$g \leq f \leq h \text{ on } A,$$

such that

$$\overline{\int}_A f = \overline{\int}_A h \text{ and } \underline{\int}_A f = \underline{\int}_A g.$$

We can take $g, h \geq 0$ if $f \geq 0$ on A.

Proof. If

$$\overline{\int}_A f = \infty,$$

the constant map $h = \infty$ satisfies the statement of the theorem.

If

$$-\infty < \overline{\int}_A f < \infty,$$

let

$$q_n = \overline{\int}_A f + \frac{1}{n}, \quad n = 1, 2, \ldots;$$

so

$$q_n \to \overline{\int}_A f < q_n.$$

By Lemma 1, for each n there is an elementary and (extended) real (hence *measurable*) map $h_n \geq f$ on A, with

$$q_n \geq \int_A h_n \geq \overline{\int}_A f.$$

Let

$$h = \inf_n h_n \geq f.$$

By Lemma 1 in §2, h is $\mathcal{M}$-measurable on A. Also,

$$(\forall\, n) \quad q_n > \int_A h_n \geq \overline{\int}_A h \geq \overline{\int}_A f$$

by Theorem 1(c). Hence

$$\overline{\int}_A f = \lim_{n\to\infty} q_n \geq \overline{\int}_A h \geq \overline{\int}_A f,$$

so

$$\overline{\int}_A f = \overline{\int}_A h,$$

as required.

Finally, if

$$\overline{\int}_A f = -\infty,$$

the same proof works with $q_n = -n$. (Verify!)

Similarly, one finds a measurable map $g \leq f$, with

$$\underline{\int}_A f = \underline{\int}_A g. \quad \square$$

Proof of Theorem 1(h). If $f \geq 0$, choose $h \geq f$ as in Lemma 2. Let

$$D = A(h > 0) \text{ and } A_n = A\Big(h > \frac{1}{n}\Big);$$

so

$$D = \bigcup_{n=1}^{\infty} A_n \text{ (why?)},$$

and D, $A_n \in \mathcal{M}$ by Theorem 1 of §2. Also,

$$0 = \overline{\int_A} f = \overline{\int_A} h \geq \int_{A_n} \left(\frac{1}{n}\right) = \frac{1}{n} mA_n \geq 0.$$

Thus $(\forall n)\ mA_n = 0$. Hence

$$mD = m \bigcup_{n=1}^{\infty} A_n = mA(h > 0) = 0;$$

so $0 \leq f \leq h \leq 0$ (i.e., $f = 0$) a.e. on A.

The case $f \leq 0$ reduces to $(-f) \geq 0$. $\square$

Corollary 1. *If*

$$\overline{\int_A} |f| < \infty,^4$$

then $|f| < \infty$ *a.e. on* A, *and* $A(f \neq 0)$ *is* σ*-finite.*

Proof. By Lemma 1, fix an elementary and nonnegative $h \geq |f|$ with

$$\int_A h < \infty$$

(so h is *elementary and integrable*).

Now, by Corollary 1(i)–(iii) in §4, our assertions apply to h, hence certainly to f. $\square$

Theorem 2 (additivity). *Given* $f\colon S \to E^*$ *and an* $\mathcal{M}$*-partition* $\mathcal{P} = \{B_n\}$ *of* $A \in \mathcal{M}$, *we have*

$$\text{(2)} \qquad \text{(a)} \ \overline{\int_A} f = \sum_n \overline{\int_{B_n}} f \quad \textit{and} \quad \text{(b)} \ \underline{\int_A} f = \sum_n \underline{\int_{B_n}} f,$$

provided

$$\overline{\int_A} f \quad \left(\underline{\int_A} f, \ \textit{respectively}\right)$$

is orthodox, or $\mathcal{P}$ *is finite.*

Hence if f *is integrable on each of* finitely *many disjoint* $\mathcal{M}$*-sets* B_n, *it is so on*

$$A = \bigcup_n B_n,$$

and formulas (2)(a)(b) *apply.*

---

[4] It suffices that f be *integrable* on A (apply the same proof to f^+ and f^-).

Proof. Assume first $f \geq 0$ on A. Then by Theorem 1(f), if one of

$$\overline{\int}_{B_n} f = \infty,$$

so is $\overline{\int}_A f$, and all is trivial. Thus assume all $\int_{B_n} f$ are finite.

Then for any $\varepsilon > 0$ and $n \in N$, there is an elementary and nonnegative map $h_n \geq f$ on B_n, with

$$\int_{B_n} h_n < \overline{\int}_{B_n} f + \frac{\varepsilon}{2^n}.$$

(Why?) Now define $h\colon A \to E^*$ by $h = h_n$ on B_n, $n = 1, 2, \ldots$.

Clearly, h is elementary and nonnegative on each B_n, hence on A (Corollary 3 in §1), and $h \geq f$ on A. Thus by Theorem 1 of §4,

$$\overline{\int}_A f \leq \int_A h = \sum_n \int_{B_n} h_n \leq \sum_n \left(\overline{\int}_{B_n} f + \frac{\varepsilon}{2^n} \right) \leq \sum_n \overline{\int}_{B_n} f + \varepsilon.$$

Making $\varepsilon \to 0$, we get

$$\overline{\int}_A f \leq \sum_n \overline{\int}_{B_n} f.$$

To prove also

$$\overline{\int}_A f \geq \sum_n \overline{\int}_{B_n} f,$$

take any elementary and nonnegative map $H \geq f$ on A. Then again,

$$\int_A H = \sum_n \int_{B_n} H \geq \sum_n \overline{\int}_{B_n} f.$$

As this holds for *any* such H, we also have

$$\overline{\int}_A f = \inf_H \int_A H \geq \sum_n \overline{\int}_{B_n} f.$$

This proves formula (a) for $f \geq 0$. The proof of (b) is quite similar.

If $f \ngeq 0$, we have

$$\overline{\int}_A f = \overline{\int}_A f^+ - \underline{\int}_A f^-,$$

where by the first part of the proof,

$$\overline{\int}_A f^+ = \sum_n \overline{\int}_{B_n} f^+ \text{ and } \underline{\int}_A f^- = \sum_n \underline{\int}_{B_n} f^-.$$

If

$$\overline{\int}_A f$$

is orthodox, one of these sums must be finite, and so their difference may be rearranged to yield

$$\overline{\int}_A f = \sum_n \Bigl(\overline{\int}_{B_n} f^+ - \underline{\int}_{B_n} f^-\Bigr) = \sum_n \overline{\int}_{B_n} f,$$

proving (a). Similarly for (b).

This rearrangement works also if $\mathcal{P}$ is *finite* (i.e., the sums have a finite number of terms). For, then, all reduces to commutativity and associativity of addition, and our conventions (2*) of Chapter 4, §4. Thus all is proved. □

Corollary 2. *If* $mQ = 0$ $(Q \in \mathcal{M})$, *then for* $A \in \mathcal{M}$

$$\overline{\int}_{A-Q} f = \overline{\int}_A f \text{ and } \underline{\int}_{A-Q} f = \underline{\int}_A f.$$

For by Theorem 2,

$$\overline{\int}_A f = \overline{\int}_{A-Q} f + \overline{\int}_{A\cap Q} f,$$

where

$$\overline{\int}_{A\cap Q} f = 0$$

by Theorem 1(b).

Corollary 3. *If*

$$\overline{\int}_A f \ \Bigl(\text{or } \underline{\int}_A f\Bigr)$$

is orthodox, so is

$$\overline{\int}_X f \ \Bigl(\underline{\int}_X f\Bigr)$$

whenever $A \supseteq X$, $X \in \mathcal{M}$.

For if

$$\overline{\int}_A f^+, \overline{\int}_A f^-, \underline{\int}_A f^+, \text{ or } \underline{\int}_A f^- \text{ is finite,}$$

it remains so also when A is reduced to X (see Theorem 1(f)). Hence orthodoxy follows by formula (1).

Note 4. Given $f\colon S \to E^*$, we can define two additive (by Theorem 2) set functions $\overline{s}$ and $\underline{s}$ by setting for $X \in \mathcal{M}$

$$\overline{s}X = \overline{\int}_X f \text{ and } \underline{s}X = \underline{\int}_X f.$$

They are called, respectively, the *upper* and lower indefinite integrals of f, also denoted by

$$\overline{\int} f \text{ and } \underline{\int} f$$

(or $\overline{s}_f$ and $\underline{s}_f$).

By Theorem 2 and Corollary 3, if

$$\overline{\int}_A f$$

is orthodox, then $\overline{s}$ is σ-additive (and semifinite) when restricted to $\mathcal{M}$-sets $X \subseteq A$. Also,

$$\overline{s}\emptyset = \underline{s}\emptyset = 0$$

by Theorem 1(b).

Such set functions are called *signed measures* (see Chapter 7, §11). In particular, if $f \geq 0$ on S, $\overline{s}$ and $\underline{s}$ are σ-additive and nonnegative on all of $\mathcal{M}$, hence *measures* on $\mathcal{M}$.

Theorem 3. *If $f\colon S \to E^*$ is m-measurable (Definition 2 in §3) on A, then*

$$\overline{\int}_A f = \underline{\int}_A f.$$

Proof. First, let $f \geq 0$ on A. By Corollary 2, we may assume that f is $\mathcal{M}$-measurable on A (drop a set of measure zero). Now fix $\varepsilon > 0$.

Let $A_0 = A(f = 0)$, $A_\infty = A(f = \infty)$, and

$$A_n = A\big((1+\varepsilon)^n \leq f < (1+\varepsilon)^{n+1}\big), \quad n = 0, \pm 1, \pm 2, \ldots .$$

Clearly, these are disjoint $\mathcal{M}$-sets (Theorem 1 of §2), and

$$A = A_0 \cup A_\infty \cup \bigcup_{n=-\infty}^{\infty} A_n.$$

Thus, setting

$$g = \begin{cases} 0 & \text{on } A_0, \\ \infty & \text{on } A_\infty, \text{ and} \\ (1+\varepsilon)^n & \text{on } A_n \ (n = 0, \pm 1, \pm 2, \ldots) \end{cases}$$

and

$$h = (1 + \varepsilon)\, g \text{ on } A,$$

we obtain two elementary and nonnegative maps, with

$$g \le f \le h \text{ on } A. \text{ (Why?)}$$

By Note 1,

$$\underline{\int}_A g = \overline{\int}_A g.$$

Now, if $\int_A g = \infty$, then

$$\overline{\int}_A f \ge \underline{\int}_A f \ge \int_A g$$

yields

$$\overline{\int}_A f \ge \underline{\int}_A f = \infty.$$

If, however, $\int_A g < \infty$, then

$$\int_A h = \int_A (1+\varepsilon) g = (1+\varepsilon) \int_A g < \infty;$$

so g and h are *elementary and integrable* on A. Thus by Theorem 2(ii) in §4,

$$\int_A h - \int_A g = \int_A (h - g) = \int_A ((1+\varepsilon)g - g) = \varepsilon \int_A g.$$

Moreover, $g \le f \le h$ implies

$$\int_A g \le \underline{\int}_A f \le \overline{\int}_A f \le \int_A h;$$

so

$$\left| \overline{\int}_A f - \underline{\int}_A f \right| \le \int_A h - \int_A g \le \varepsilon \int_A g.$$

As ε is arbitrary, all is proved for $f \ge 0$.

The case $f \ngeq 0$ now follows by formula (1), since f^+ and f^- are $\mathcal{M}$-*measurable* (Theorem 2 in §2). $\square$

Problems on Integration of Extended-Real Functions

1. Using the formulas in (1) and our conventions, verify that

(i) $\overline{\int}_A f = +\infty$ iff $\overline{\int}_A f^+ = \infty$;

(ii) $\underline{\int}_A f = \infty$ iff $\underline{\int}_A f^+ = \infty$; and

(iii) $\overline{\int}_A f = -\infty$ iff $\underline{\int}_A f^- = \infty$ and $\overline{\int}_A f^+ < \infty$.

(iv) Derive a condition similar to (iii) for $\underline{\int}_A f = -\infty$.

(v) Review Problem 6 of Chapter 4, §4.

2. Fill in the missing proof details in Theorems 1 to 3 and Lemmas 1 and 2.

3. Prove that if $\underline{\int}_A f = \infty$, there is an *elementary and (extended) real* map $g \leq f$ on A, with $\int_A g = \infty$.

[Outline: By Problem 1, we have

$$\underline{\int}_A f^+ = \infty.$$

As Lemmas 1 and 2 surely hold for *nonnegative* functions, fix a *measurable* $F \leq f^+$ $(F \geq 0)$, with

$$\int_A F = \underline{\int}_A f^+ = \infty.$$

Arguing as in Theorem 3, find an elementary and nonnegative map $g \leq F$, with

$$(1+\varepsilon)\int_A g = \int_A F = \infty;$$

so $\int_A g = \infty$ and $0 \leq g \leq F \leq f^+$ on A.

Let

$$A_+ = A(F > 0) \in \mathcal{M}$$

and

$$A_0 = A(F = 0) \in \mathcal{M}$$

(Theorem 1 in §2). On A_+,

$$g \leq F \leq f^+ = f \text{ (why?)},$$

while on A_0, $g = F = 0$; so

$$\int_{A_+} g = \int_A g = \infty \text{ (why?)}.$$

Now *redefine* $g = -\infty$ on A_0 (only). Show that g is then the required function.]

4. For *any* $f\colon S \to E^*$, prove the following.

(a) If $\overline{\int}_A f < \infty$, then $f < \infty$ a.e. on A.

(b) If $\underline{\int}_A f$ is orthodox and $> -\infty$, then $f > -\infty$ a.e. on A.

[Hint: Use Problem 1 and apply Corollary 1 to f^+; thus prove (a). Then for (b), use Theorem 1(e′).]

⇒5. For *any* $f, g\colon S \to E^*$, prove that

(i) $\overline{\int}_A f + \overline{\int}_A g \geq \overline{\int}_A (f+g)$, and

(ii) $\underline{\int}_A (f+g) \geq \underline{\int}_A f + \underline{\int}_A g$ if $\left|\underline{\int}_A g\right| < \infty$.

[Hint: Suppose that

$$\overline{\int}_A f + \overline{\int}_A g < \overline{\int}_A (f+g).$$

Then there are numbers

$$u > \overline{\int}_A f \text{ and } v > \overline{\int}_A g,$$

with

$$u + v \leq \overline{\int}_A (f+g).$$

(Why?) Thus Lemma 1 yields *elementary and (extended) real* maps $F \geq f$ and $G \geq g$ such that

$$u > \overline{\int}_A F \text{ and } v > \overline{\int}_A G.$$

As $f + g \leq F + G$ on A, Theorem 1(c) of §5 and Problem 6 of §4 show that

$$\overline{\int}_A (f+g) \leq \int_A (F+G) = \int_A F + \int_A G < u + v,$$

contrary to

$$u + v \leq \overline{\int}_A (f+g).$$

Similarly prove clause (ii).]

6. Continuing Problem 5, prove that

$$\overline{\int}_A (f+g) \geq \overline{\int}_A f + \underline{\int}_A g \geq \underline{\int}_A (f+g) \geq \underline{\int}_A f + \underline{\int}_A g,$$

provided $\left|\underline{\int}_A g\right| < \infty$.

[Hint for the second inequality: We may assume that

$$\overline{\int}_A (f+g) < \infty \text{ and } \overline{\int}_A f > -\infty.$$

(Why?) Apply Problems 5 and 4(a) to

$$\overline{\int}_A ((f+g) + (-g)).$$

Use Theorem 1(e′).]

7. Prove the following.

(i) $\overline{\int}_A |f| < \infty$ iff $-\infty < \underline{\int}_A f \leq \overline{\int}_A f < \infty$.

(ii) If $\overline{\int}_A |f| < \infty$ and $\overline{\int}_A |g| < \infty$, then

$$\left| \overline{\int}_A f - \overline{\int}_A g \right| \le \overline{\int}_A |f - g|$$

and

$$\left| \underline{\int}_A f - \underline{\int}_A g \right| \le \overline{\int}_A |f - g|.$$

[Hint: Use Problems 5 and 6.]

8. Show that any signed measure $\overline{s}_f$ (Note 4) is the difference of two *measures*: $\overline{s}_f = \overline{s}_{f^+} - \overline{s}_{f^-}$.

§6. Integrable Functions. Convergence Theorems

I. Some important theorems apply to *integrable* functions.

Theorem 1 (linearity of the integral)**.** *If $f, g \colon S \to E^*$ are integrable on a set $A \in \mathcal{M}$ in $(S, \mathcal{M}, m)$, so is*

$$pf + qg$$

for any $p, q \in E^1$, and

$$\int_A (pf + qg) = p \int_A f + q \int_A g;$$

in particular,

$$\int_A (f \pm g) = \int_A f \pm \int_A g.$$

Proof. By Problem 5 in §5,

$$\overline{\int}_A f + \overline{\int}_A g \ge \overline{\int}_A (f+g) \ge \underline{\int}_A (f+g) \ge \underline{\int}_A f + \underline{\int}_A g.$$

(Here

$$\overline{\int}_A f,\ \underline{\int}_A f,\ \overline{\int}_A g, \text{ and } \underline{\int}_A g$$

are *finite* by integrability; so all is orthodox.)

As

$$\overline{\int}_A f = \underline{\int}_A f \text{ and } \overline{\int}_A g = \underline{\int}_A g,$$

the inequalities turn into equalities, so that

$$\int_A f + \int_A g = \overline{\int}_A (f+g) = \underline{\int}_A (f+g).$$

Using also Theorem 1(e)(e′) from §5, we obtain the desired result for any $p, q \in E^1$. $\square$

Theorem 2. *A function* $f\colon S \to E^*$ *is integrable on* A *in* $(S, \mathcal{M}, m)$ *iff*

(i) *it is* m*-measurable on* A, *and*

(ii) $\overline{\int}_A f$ *(equivalently* $\overline{\int}_A |f|$*) is finite.*

Proof. If these conditions hold, f is integrable on A by Theorem 3 of §5.

Conversely, let

$$\overline{\int}_A f = \underline{\int}_A f \neq \pm\infty.$$

Using Lemma 2 in §5, fix *measurable* maps g and h $(g \le f \le h)$ on A, with

$$\int_A g = \int_A f = \int_A h \neq \pm\infty.$$

By Theorem 3 in §5, g and h are integrable on A; so by Theorem 1,

$$\int_A (h-g) = \int_A h - \int_A g = 0.$$

As

$$h - g \ge h - f \ge 0,$$

we get

$$\int_A (h-f) = 0,$$

and so by Theorem 1(h) of §5, $h - f = 0$ a.e. on A.

Hence f is *almost* measurable on A, and

$$\int_A f \neq \pm\infty$$

by assumption. From formula (1), we then get

$$\int_A f^+ \text{ and } \int_A f^- < \infty,$$

and hence

$$\int_A |f| = \int_A (f^+ + f^-) = \int_A f^+ + \int_A f^- < \infty$$

by Theorem 1 and by Theorem 2 of §2. Thus all is proved. $\square$

Simultaneously, we also obtain the following corollary.

Corollary 1. *A function* $f\colon S \to E^*$ *is integrable on* A *iff* f^+ *and* f^- *are.*

Corollary 2. *If* $f, g\colon S \to E^*$ *are integrable on* A*, so also are*

$$f \vee g,\ f \wedge g,\ |f|,\ \text{and } kf \text{ for } k \in E^1,$$

with

$$\int_A kf = k \int_A f.$$

Exercise!

For *products* fg, this holds if f or g is *bounded.* In fact, we have the following theorem.

Theorem 3 (weighted law of the mean)**.** *Let* f *be* m*-measurable and bounded on* A*. Set*

$$p = \inf f[A] \text{ and } q = \sup f[A].$$

Then if g *is* m*-integrable on* A*, so is* fg*, and*

$$\int_A f|g| = c \int_A |g|$$

for some $c \in [p, q]$*.*

If, further, f *also has the Darboux property on* A *(Chapter 4, §9), then* $c = f(x_0)$ *for some* $x_0 \in A$*.*

Proof. By assumption,

$$(\exists\, k \in E^1) \quad |f| \le k$$

on A. Hence if $\int_A |g| = 0$,

$$\left|\int_A f|g|\right| \le \int_A |fg| \le k \int_A |g| = 0;$$

so *any* $c \in [p, q]$ yields

$$\int_A f|g| = c \int_A |g| = 0.$$

If, however, $\int_A |g| \ne 0$, the number

$$c = \left(\int_A f|g|\right) \bigg/ \int_A |g|$$

is the required constant.

Moreover, as f and g are m-measurable on A, so is fg; and as

$$\left|\int_A fg\right| \le |c| \int_A |g| < \infty,$$

fg is integrable on A by Theorem 2.

Finally, if f has the Darboux property and if $p < c < q$ (with p, q as above), then

$$f(x) < c < f(y)$$

for some $x, y \in A$ (why?); hence by the Darboux property, $f(x_0) = c$ for some $x_0 \in A$.

If, however,

$$c \leq \inf f[A] = p,$$

then

$$(f - c)|g| \geq 0$$

and

$$\int_A (f - c)|g| = \int_A f|g| - c \int_A |g| = 0 \text{ (why?);}$$

so by Theorem 1(h) in §5, $f - c = 0$ a.e. on A. Then surely $f(x_0) = c$ for some $x_0 \in A$ (except the trivial case $mA = 0$). This also implies $c \in f[A] \in [p, q]$.

Proceed similarly in the case $c \geq q$. □

Corollary 3. *If f is integrable on $A \in \mathcal{M}$, it is so on any $B \subseteq A$ $(B \in \mathcal{M})$.*

Proof. Apply Theorem 1(f) in §5, and Theorem 3 of §5, to f^+ and f^-. □

II. Convergence Theorems. If $f_n \to f$ on A (pointwise, a.e., or uniformly), does it follow that

$$\int_A f_n \to \int_A f?$$

To give some answers, we need a lemma.

Lemma 1. *If $f \geq 0$ on $A \in \mathcal{M}$ and if*

$$\underline{\int}_A f > p \in E^*,$$

there is an elementary and nonnegative map g on A such that

$$\int_A g > p,$$

and $g < f$ on A except only at those $x \in A$ (if any) at which

$$f(x) = g(x) = 0.$$

(We then briefly write $g \subset f$ on A.)

Proof. By Lemma 1 in §5, there is an elementary and nonnegative map $G \leq f$ on A, with

$$\underline{\int_A} f \geq \int_A G > p.$$

For the rest, proceed as in Problem 7 of §4, replacing f by G there. □

Theorem 4 (monotone convergence)**.** *If* $0 \leq f_n \nearrow f$ *(a.e.) on* $A \in \mathcal{M}$, *i.e.,*

$$0 \leq f_n \leq f_{n+1} \quad (\forall\, n),$$

and $f_n \to f$ *(a.e.) on* A, *then*

$$\overline{\int}_A f_n \nearrow \overline{\int}_A f.$$

Proof for $\mathcal{M}$-measurable f_n and f on A.[1] By Corollary 2 in §5, we may assume that $f_n \nearrow f$ (pointwise) on A (otherwise, drop a null set).

By Theorem 1(c) of §5, $0 \leq f_n \nearrow f$ implies

$$0 \leq \int_A f_n \leq \int_A f,$$

and so

$$\lim_{n\to\infty} \int_A f_n \leq \int_A f.$$

The limit, call it p, *exists* in E^*, as $\{\int_A f_n\}\uparrow$. It remains to show that

$$p \geq \overline{\int}_A f = \underline{\int_A} f.$$

(We know that

$$\overline{\int}_A f = \underline{\int_A} f,$$

by the assumed measurability of f; see Theorem 3 in §5.)

Suppose

$$\underline{\int_A} f > p.$$

Then Lemma 1 yields an elementary and nonnegative map $g \subset f$ on A, with

$$p < \int_A g.$$

Let

$$A_n = A(f_n \geq g), \quad n = 1, 2, \ldots .$$

---

[1] For the general case, see Problem 5.

Then $A_n \in \mathcal{M}$ and

$$A_n \nearrow A = \bigcup_{n=1}^{\infty} A_n.$$

For if $f(x) = 0$, then $x \in A_1$, and if $f(x) > 0$, then $f(x) > g(x)$, so that $f_n(x) > g(x)$ for large n; hence $x \in A_n$.

By Note 4 in §5, the set function $s = \int g$ is a *measure*, hence continuous by Theorem 2 in Chapter 7, §4. Thus

$$\int_A g = sA = \lim_{n\to\infty} sA_n = \lim_{n\to\infty} \int_{A_n} g.$$

But as $g \le f_n$ on A_n, we have

$$\int_{A_n} g \le \int_{A_n} f_n \le \int_A f_n.$$

Hence

$$\int_A g = \lim \int_{A_n} g \le \lim \int_A f_n = p,$$

contrary to $p < \int_A g$. This contradiction completes the proof. □

Lemma 2 (Fatou). *If* $f_n \ge 0$ *on* $A \in \mathcal{M}$ $(n = 1, 2, \dots)$, *then*

$$\overline{\int}_A \underline{\lim}\, f_n \le \underline{\lim} \overline{\int}_A f_n.$$

Proof. Let

$$g_n = \inf_{k \ge n} f_k, \quad n = 1, 2, \dots;$$

so $f_n \ge g_n \ge 0$ and $\{g_n\}\uparrow$ on A. Thus by Theorem 4,

$$\overline{\int}_A \lim g_n = \lim \overline{\int}_A g_n = \underline{\lim} \overline{\int}_A g_n \le \underline{\lim} \overline{\int}_A f_n.$$

But

$$\lim_{n\to\infty} g_n = \sup_n g_n = \sup_n \inf_{k \ge n} f_k = \underline{\lim}\, f_n.$$

Hence

$$\overline{\int}_A \underline{\lim}\, f_n = \overline{\int}_A \lim g_n \le \underline{\lim} \overline{\int}_A f_n,$$

as claimed. □

Theorem 5 (dominated convergence). *Let* $f_n\colon S \to E$ *be* m*-measurable on* $A \in \mathcal{M}$ $(n = 1, 2, \dots)$. *Let*

$$f_n \to f \text{ (a.e.) on } A.$$

Then

$$\lim_{n\to\infty} \int_A |f_n - f| = 0,$$

provided that there is a map $g\colon S \to E^1$ *such that*

$$\int_A g < \infty$$

and

$$(\forall\, n) \quad |f_n| \le g \text{ a.e. on } A.$$

Proof. Neglecting null sets, we may assume that

$$|f_n| \le g < \infty$$

on A and $f_n \to f$ (*pointwise*) on A; so $|f| \le g$ and

$$|f_n - f| \le |f_n| + |f| \le 2g$$

on A. As $|f| < \infty$, we have

$$|f_n - f| \to 0$$

on A. Hence, setting

$$h_n = 2g - |f_n - f| \ge 0,$$

we get

$$2g = \lim_{n\to\infty} h_n = \underline{\lim}\, h_n.$$

We may also assume that g is measurable on A. (If not, replace it by a measurable $G \ge g$, with

$$\int_A G = \int_A g < \infty,$$

by Lemma 2 in §5.) Then all

$$h_n = 2g - |f_n - f|$$

are measurable (even integrable) on A.

Thus by Lemma 2,

$$\begin{aligned}\int_A 2g = \int_A \underline{\lim}\, h_n \le \underline{\lim} \int_A (2g - |f_n - f|) \\ = \underline{\lim}\Big(\int_A 2g + \int_A (-|f_n - f|)\Big) \\ = \int_A 2g + \underline{\lim}\Big(-\int_A |f_n - f|\Big) \\ = \int_A 2g - \overline{\lim} \int_A |f_n - f|.\end{aligned}$$

(See Problems 5 and 8 in Chapter 2, §13.)

Canceling $\int_A 2g$ (finite!), we have

$$0 \le -\overline{\lim} \int_A |f_n - f|.$$

Hence

$$0 \ge \overline{\lim} \int_A |f_n - f| \ge \underline{\lim} \int_A |f_n - f| \ge 0,$$

as $|f_n - f| \ge 0$. This yields

$$0 = \overline{\lim} \int_A |f_n - f| = \underline{\lim} \int_A |f_n - f| = \lim \int_A |f_n - f|,$$

as required. □

Note 1. Theorem 5 holds also for complex and vector-valued functions (for $|f_n - f|$ is *real*).

In the extended-real case, Theorems 1(g) in §5 and Theorems 1 and 2 in §6 yield

$$\left|\int_A f_n - \int_A f\right| = \left|\int_A (f_n - f)\right| \le \int_A |f_n - f| \to 0,$$

i.e.,

$$\int_A f_n \to \int_A f.$$

Moreover, f is *integrable* on A, being measurable (why?), with

$$\int_A |f| \le \int_A g < \infty.$$

For complex and vector-valued functions, this will follow from §7. Observe that Theorem 5, *unlike Theorem* 4, requires the m-measurability of the f_n.

Note 2. Theorem 5 fails if there is no "*dominating*"

$$g \geq |f_n| \text{ with } \int_A g < \infty,$$

even if f and the f_n are integrable.

Example.

Let m be Lebesgue measure in $A = E^1$, $f = 0$, and

$$f_n = \begin{cases} 1 & \text{on } [n, n+1], \\ 0 & \text{elsewhere.} \end{cases}$$

Then $f_n \to f$ and $\int_A f_n = 1$; so

$$\lim_{n \to \infty} \int_A f_n = 1 \neq 0 = \int_A f.$$

The trouble is that any

$$g \geq f_n \quad (n = 1, 2, \ldots)$$

would have to be ≥ 1 on $B = [1, \infty)$; so

$$\int_A g \geq \int_B g = 1 \cdot mB = \infty,$$

instead of $\int_A g < \infty$.

This example also shows that $f_n \to f$ *alone* does not imply

$$\int_A f_n \to \int_A f.$$

Theorem 6 (absolute continuity of the integral)**.** *Given* $f\colon S \to E$ *with*

$$\overline{\int}_A |f| < \infty$$

and $\varepsilon > 0$, *there is* $\delta > 0$ *such that*

$$\overline{\int}_X |f| < \varepsilon$$

whenever

$$mX < \delta \quad (A \supseteq X,\ X \in \mathcal{M}).$$

Proof. By Lemma 2 in §5, fix $h \geq |f|$, *measurable* on A, with

$$\int_A h = \overline{\int}_A |f| < \infty.$$

Neglecting a null set, we assume that $|h| < \infty$ on A (Corollary 1 of §5). Now, $(\forall\, n)$ set

$$g_n(x) = \begin{cases} h(x), & x \in A_n = A(h < n), \\ 0, & x \in -A_n. \end{cases}$$

Then $g_n \le n$ and g_n is measurable on A. (Why?)

Also, $g_n \ge 0$ and $g_n \to h$ (*pointwise*) on A.

For let $\varepsilon > 0$, fix $x \in A$, and find $k > h(x)$. Then

$$(\forall\, n \ge k) \quad h(x) \le n \text{ and } g_n(x) = h(x).$$

So

$$(\forall\, n \ge k) \quad |g_n(x) - h(x)| = 0 < \varepsilon.$$

Clearly, $g_n \le h$. Hence by Theorem 5

$$\lim_{n\to\infty} \int_A |h - g_n| = 0.$$

Thus we can fix n so large that

$$\int_A (h - g_n) < \frac{1}{2}\varepsilon.$$

For *that* n, let

$$\delta = \frac{\varepsilon}{2n}$$

and take any $X \subseteq A$ $(X \in \mathcal{M})$, with $mX < \delta$.

As $g_n \le n$ (see above), Theorem 1(c) in §5 yields

$$\int_X g_n \le \int_X (n) = n \cdot mX < n\delta = \frac{1}{2}\varepsilon.$$

Hence as $|f| \le h$ and

$$\int_X (h - g_n) \le \int_A (h - g_n) < \frac{1}{2}\varepsilon$$

(Theorem 1(f) of §5), we obtain

$$\overline{\int}_X |f| \le \int_X h = \int_X (h - g_n) + \int_X g_n < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon,$$

as required. $\square$

Problems on Integrability and Convergence Theorems

1. Fill in the missing details in the proofs of this section.

2. (i) Show that if $f\colon S \to E^*$ is bounded and m-measurable on A, with $mA < \infty$, then f is m-integrable on A (Theorem 2) and

$$\int_A f = c \cdot mA,$$

where $\inf f[A] \le c \le \sup f[A]$.

(ii) Prove that if f also has the Darboux property on A, then

$$(\exists\, x_0 \in A) \quad c = f(x_0).$$

[Hint: Take $g = 1$ in Theorem 3.]

(iii) What results if $A = [a, b]$ and $m =$ Lebesgue measure?

3. Prove Theorem 4 assuming that the f_n are measurable on A and that

$$(\exists\, k) \quad \int_A f_k > -\infty$$

instead of $f_n \ge 0$.

[Hint: As $\{f_n\}\uparrow$, show that

$$(\forall\, n \ge k) \quad \int_A f_n > -\infty.$$

If

$$(\exists\, n) \quad \int_A f_n = \infty,$$

then

$$\int_A f = \lim \int_A f_n = \infty.$$

Otherwise,

$$(\forall\, n \ge k) \quad \left|\int_A f_n\right| < \infty;$$

so f_n is *integrable*. (Why?) By Corollary 1 in §5, assume $|f_n| < \infty$. (Why?) Apply Theorem 4 to $h_n = f_n - f_k$ $(n \ge k)$, considering two cases:

$$\int_A h < \infty \text{ and } \int_A h = \infty.]$$

4. Show that if $f_n \nearrow f$ (pointwise) on $A \in \mathcal{M}$, there are $\mathcal{M}$-measurable maps $F_n \ge f_n$ and $F \ge f$ on A, with $F_n \nearrow F$ (pointwise) on A, such that

$$\int_A F = \overline{\int}_A f \text{ and } \int_A F_n = \overline{\int}_A f_n.$$

[Hint: By Lemma 2 of §5, fix measurable maps $h \geq f$ and $h_n \geq f_n$ with the *same* integrals. Let

$$F_n = \inf_{k \geq n} (h \wedge h_k), \quad n = 1, 2, \dots ,$$

and $F = \sup_n F_n \leq h$. (Why?) Proceed.]

5. For $A \in \mathcal{M}$ and any (even nonmeasurable) functions $f, f_n \colon S \to E^*$, prove the following.

(i) If $f_n \nearrow f$ (a.e.) on A, then

$$\overline{\int}_A f_n \nearrow \overline{\int}_A f,$$

provided

$$(\exists\, n) \quad \overline{\int}_A f_n > -\infty.$$

(ii) If $f_n \searrow f$ (a.e.) on A, then

$$\underline{\int}_A f_n \searrow \underline{\int}_A f,$$

provided

$$(\exists\, n) \quad \underline{\int}_A f_n < \infty.$$

[Hint: Replace f, f_n by F, F_n as in Problem 4. Then apply Problem 3 to F_n; thus obtain (i). For (ii), use (i) and Theorem 1(e′) in §5. (All is *orthodox*; why?)]

6. Show by examples that

(i) the conditions

$$\overline{\int}_A f_n > -\infty \text{ and } \underline{\int}_A f_n < \infty$$

in Problem 5 are *essential*; and

(ii) Problem 5(i) fails for *lower* integrals. What about 5(ii)?

[Hints: (i) Let $A = (0, 1) \subset E^1$, $m =$ Lebesgue measure, $f_n = -\infty$ on $(0, \frac{1}{n})$, $f_n = 1$ elsewhere.

(ii) Let $\mathcal{M} = \{E^1, \emptyset\}$, $mE^1 = 1$, $m\emptyset = 0$, $f_n = 1$ on $(-n, n)$, $f_n = 0$ elsewhere. If $f = 1$ on $A = E^1$, then $f_n \to f$, but not

$$\underline{\int}_A f_n \to \underline{\int}_A f.$$

Explain!]

7. Given $f_n \colon S \to E^*$ and $A \in \mathcal{M}$, let

$$g_n = \inf_{k \geq n} f_k \text{ and } h_n = \sup_{k \geq n} f_k \quad (n = 1, 2, \dots).$$

Prove that

(i) $\overline{\int}_A \underline{\lim}\, f_n \le \underline{\lim}\, \overline{\int}_A f_n$ provided $(\exists\, n)\ \overline{\int}_A g_n > -\infty$; and

(ii) $\underline{\int}_A \overline{\lim}\, f_n \le \overline{\lim}\, \underline{\int}_A f_n$ provided $(\exists\, n)\ \underline{\int}_A h_n < \infty$.
[Hint: Apply Problem 5 to g_n and h_n.]

(iii) Give examples for which

$$\overline{\int}_A \underline{\lim}\, f_n \neq \underline{\lim}\, \overline{\int}_A f_n \text{ and } \underline{\int}_A \overline{\lim}\, f_n \neq \overline{\lim}\, \underline{\int}_A f_n.$$

(See Note 2).

8. Let $f_n \ge 0$ on $A \in \mathcal{M}$ and $f_n \to f$ (a.e.) on A. Let $A \supseteq X$, $X \in \mathcal{M}$. Prove the following.

(i) If

$$\overline{\int}_A f_n \to \overline{\int}_A f < \infty,$$

then

$$\overline{\int}_X f_n \to \overline{\int}_X f.$$

(ii) This fails for *sign-changing* f_n.

[Hints: If (i) *fails*, then

$$\underline{\lim}\, \overline{\int}_X f_n < \overline{\int}_X f \text{ or } \underline{\lim}\, \overline{\int}_X f_n > \overline{\int}_X f.$$

Find a subsequence of

$$\left\{\overline{\int}_X f_n\right\} \text{ or } \left\{\overline{\int}_{A-X} f_n\right\}$$

contradicting Lemma 2.

(ii) Let $m =$ Lebesgue measure; $A = (0, 1)$, $X = (0, \frac{1}{2})$,

$$f_n = \begin{cases} n & \text{on } \left(0, \frac{1}{2n}\right], \\ -n & \text{on } \left(1 - \frac{1}{2n}, 1\right).\end{cases}]$$

⇒9. (i) Show that if f and g are m-measurable and nonnegative on A, then

$$(\forall\, a, b \ge 0) \quad \int_A (af + bg) = a\int_A f + b\int_A g.$$

(ii) If, in addition, $\int_A f < \infty$ or $\int_A g < \infty$, this formula holds for *any* $a, b \in E^1$.

[Hint: Proceed as in Theorem 1.]

$\Rightarrow$**10.** If

$$f = \sum_{n=1}^{\infty} f_n,$$

with all f_n measurable and nonnegative on A, then

$$\int_A f = \sum_{n=1}^{\infty} \int_A f_n.$$

[Hint: Apply Theorem 4 to the maps

$$g_n = \sum_{k=1}^{n} f_k \nearrow f.$$

Use Problem 9.]

11. If

$$q = \sum_{n=1}^{\infty} \int_A |f_n| < \infty$$

and the f_n are m-measurable on A, then

$$\sum_{n=1}^{\infty} |f_n| < \infty \ (a.e.) \text{ on } A$$

and $f = \sum_{n=1}^{\infty} f_n$ is m-integrable on A, with

$$\int_A f = \sum_{n=1}^{\infty} \int_A f_n.$$

[Hint: Let $g = \sum_{n=1}^{\infty} |f_n|$. By Problem 10,

$$\int_A g = \sum_{n=1}^{\infty} \int_A |f_n| = q < \infty;$$

so $g < \infty$ $(a.e.)$ on A. (Why?) Apply Theorem 5 and Note 1 to the maps

$$g_n = \sum_{k=1}^{n} f_k;$$

note that $|g_n| \leq g$.]

12. (Convergence in measure; see Problem 11(ii) of §3).

(i) Prove *Riesz' theorem*: If $f_n \to f$ in measure on $A \subseteq S$, there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ (almost uniformly), hence (a.e.), on A.
[Outline: Taking

$$\sigma_k = \delta_k = 2^{-k},$$

pick, step by step, naturals

$$n_1 < n_2 < \cdots < n_k < \cdots$$

and sets $D_k \in \mathcal{M}$ such that $(\forall\, k)$

$$mD_k < 2^{-k}$$

and

$$\rho'(f_{n_k}, f) < 2^{-k}$$

on $A - D_k$. (Explain!) Let

$$E_n = \bigcup_{k=n}^{\infty} D_k,$$

$mE_n < 2^{1-n}$. (Why?) Show that

$$(\forall\, n)\ (\forall\, k > n) \quad \rho'(f_{n_k}, f) < 2^{1-n}$$

on $A - E_n$. Use Problem 11 in §3.]

(ii) For maps $f_n \colon S \to E$ and $g \colon S \to E^1$ deduce that if

$$f_n \to f$$

in measure on A and

$$(\forall\, n) \quad |f_n| \le g \text{ (a.e.) on } A,$$

then

$$|f| \le g \text{ (a.e.) on } A.$$

[Hint: $f_{n_k} \to f$ *(a.e.)* on A.]

13. Continuing Problem 12(ii), let

$$f_n \to f$$

in measure on $A \in \mathcal{M}$ $(f_n \colon S \to E)$ and

$$(\forall\, n) \quad |f_n| \le g \text{ (a.e.) on } A,$$

with

$$\overline{\int_A} g < \infty.$$

Prove that

$$\lim_{n\to\infty} \overline{\int_A} |f_n - f| = 0.$$

Does

$$\overline{\int_A} f_n \to \overline{\int_A} f?$$

[Outline: From Corollary 1 of §5, infer that $g = 0$ on $A - C$, where

$$C = \bigcup_{k=1}^{\infty} C_k \ (\textit{disjoint}),$$

$mC_k < \infty$. (We may assume g $\mathcal{M}$-measurable on A. Why?) Also,

$$\infty > \int_A g = \int_{A-C} g + \int_C g = 0 + \sum_{k=1}^{\infty} \int_{C_k} g;$$

so the series *converges*. Hence

$$(\forall \varepsilon > 0)\ (\exists p) \quad \int_A g - \varepsilon < \sum_{k=1}^{p} \int_{C_k} g = \int_H g,$$

where

$$H = \bigcup_{k=1}^{p} C_k \in \mathcal{M}$$

and $mH < \infty$. As $|f_n - f| \le 2g$ (a.e.), we get

$$\text{(1)} \quad \underline{\int}_A |f_n - f| \le \overline{\int}_A |f_n - f| \le \overline{\int}_H |f_n - f| + \int_{A-H} 2g < \overline{\int}_H |f_n - f| + 2\varepsilon.$$

(Explain!)

As $mH < \infty$, we can fix $\sigma > 0$ with

$$\sigma \cdot mH < \varepsilon.$$

Also, by Theorem 6, fix δ such that

$$2 \int_X g < \varepsilon$$

whenever $A \supseteq X$, $X \in \mathcal{M}$ and $mX < \delta$.

As $f_n \to f$ *in measure* on H, we find $\mathcal{M}$-sets $D_n \subseteq H$ such that

$$(\forall n > n_0) \quad mD_n < \delta$$

and

$$|f_n - f| < \sigma \text{ on } A_n = H - D_n.$$

(We may use the *standard* metric, as $|f|$ and $|f_n| < \infty$ a.e. Why?) Thus from (1), we get

$$\begin{aligned}
\overline{\int}_A |f_n - f| &\le \overline{\int}_H |f_n - f| + 2\varepsilon \\
&= \overline{\int}_{A_n} |f_n - f| + \overline{\int}_{D_n} |f_n - f| + 2\varepsilon \\
&< \overline{\int}_{A_n} |f_n - f| + 3\varepsilon \\
&\le \sigma \cdot mH + 3\varepsilon < 4\varepsilon
\end{aligned}$$

for $n > n_0$. (Explain!) Hence

$$\lim \overline{\int}_A |f_n - f| = 0.$$

See also Problem 7 in §5 and Note 1 of §6 (for *measurable* functions) as regards

$$\lim \overline{\int}_A f_n.]$$

14. Do Problem 12 in §3 (Lebesgue–Egorov theorems) for $T = E$, assuming

$$(\forall\, n) \quad |f_n| \le g \ (a.e.) \text{ on } A,$$

with

$$\int_A g < \infty$$

(instead of $mA < \infty$).
[Hint: With $H_i(k)$ as before, it suffices that

$$\lim_{i\to\infty} m(A - H_i(k)) = 0.$$

(Why?) Verify that

$$(\forall\, n) \quad \rho'(f_n, f) = |f_n - f| \le 2g \ (a.e.) \text{ on } A,$$

and

$$(\forall\, i, k) \quad A - H_i(k) \subseteq A\Big(2g \ge \frac{1}{k}\Big) \cup Q \ (mQ = 0).$$

Infer that

$$(\forall\, i, k) \quad m(A - H_i(k)) < \infty.$$

Now, as $(\forall\, k)\ H_i(k) \searrow \emptyset$ (why?), right continuity applies.]

§7. Integration of Complex and Vector-Valued Functions

I. First we consider functions $f\colon S \to E^n\,(C^n)$. For such functions, it is natural (and easy) to define integration "componentwise" as follows.[1]

Definition 1.

A function $f\colon S \to E^n$ is said to be *integrable* on $A \in \mathcal{M}$ iff its n (real) components, $f_1, \ldots, f_n$, are. In this case, we define

$$\int_A f = \int_A f\,dm = \Big(\int_A f_1, \int_A f_2, \ldots, \int_A f_n\Big) = \sum_{k=1}^{n} \bar{e}_k \cdot \int_A f_k, \tag{1}$$

where the $\bar{e}_k$ are basic unit vectors (as in Chapter 3, §§1–3, Theorem 2).

---

[1] As before, we presuppose an arbitrary (but fixed) measure space $(S, \mathcal{M}, m)$.

In particular, a *complex* function f is integrable on A iff its real and imaginary parts (f_{re} and f_{im}) are. Then we also say that $\int_A f$ *exists*.[2] By (1), we have

$$\int_A f = \left(\int_A f_{\text{re}}, \int_A f_{\text{im}}\right) = \int_A f_{\text{re}} + i \int_A f_{\text{im}}. \tag{2}$$

If $f\colon S \to C^n$, we use (1), with *complex* components f_k.

With this definition, integration of functions $f\colon S \to E^n\,(C^n)$ reduces to that of $f_k\colon S \to E^1\,(C)$, and one easily obtains the same theorems as in §§4–6, as far as they make sense for *vectors*.

Theorem 1. *A function $f\colon S \to E^n\,(C^n)$ is integrable on $A \in \mathcal{M}$ iff it is m-measurable on A and $\int_A |f| < \infty$.*

(Alternate definition!)

Proof. Assume the range space is E^n.

By our definition, if f is integrable on A, then its components f_k are. Thus by Theorem 2 and Corollary 1, both in §6, for $k = 1, 2, \ldots, n$, the functions f_k^+ and f_k^- are m-measurable; furthermore,

$$\int_A f_k^+ \neq \pm\infty \text{ and } \int_A f_k^- \neq \pm\infty.$$

This implies

$$\infty > \int_A f_k^+ + \int_A f_k^- = \int_A (f_k^+ + f_k^-) = \int_A |f_k|, \quad k = 1, 2, \ldots, n.$$

Since $|f|$ is m-measurable by Problem 14 in §3 ($|\cdot|$ is a *continuous* mapping from E^n to E^1), and

$$|f| = \left|\sum_{k=1}^{n} \bar{e}_k f_k\right| \leq \sum_{k=1}^{n} |\bar{e}_k|\,|f_k| = \sum_{k=1}^{n} |f_k|,$$

we get

$$\int_A |f| \leq \int_A \sum_1^n |f_k| = \sum_1^n \int_A |f_k| < \infty.$$

Conversely, if f satisfies

$$\int_A |f| < \infty$$

then

$$(\forall\, k) \quad \left|\int_A f_k\right| < \infty.$$

---

[2] For vector-valued functions, too, this phrase means *integrability*.

Also, the f_k are m-measurable if f is (see Problem 2 in §3). Hence the f_k are integrable on A (by Theorem 2 of §6), and so is f.

The proof for C^n is analogous. $\square$

Similarly for other theorems (see Problems 1 to 4 below). We have already noted that Theorem 5 of §6 holds for complex and vector-valued functions. So does Theorem 6 in §6. We prove another such proposition (Lemma 1) below.

II. Next we consider the general case, $f\colon S \to E$ (E complete). We now adopt Theorem 1 as a *definition*. (It agrees with Definition 1 of §4. Verify!) Even if $E = E^*$, we always assume $|f| < \infty$ *a.e.*; thus, dropping a null set, we can make f *finite* and use the standard metric on E^1.

First, we take up the case $mA < \infty$.

Lemma 1. *If $f_n \to f$ (uniformly) on A ($mA < \infty$), then*

$$\int_A |f_n - f| \to 0.$$

Proof. By assumption,

$$(\forall\, \varepsilon > 0)\ (\exists\, k)\ (\forall\, n > k) \quad |f_n - f| < \varepsilon \text{ on } A;$$

so

$$(\forall\, n > k) \quad \int_A |f_n - f| \le \int_A (\varepsilon) = \varepsilon \cdot mA < \infty.$$

As ε is arbitrary, the result follows. $\square$

Our goal is to prove results on *linearity* (Theorem 2) and *additivity* (Theorem 3) for *general* E; for a "limited approach," see Problem 2 for $E = E^n$ (C^n).

***Lemma 2.** *If*

$$\int_A |f| < \infty \quad (mA < \infty)$$

and

$$f = \lim_{n\to\infty} f_n \text{ (uniformly) on } A - Q \ (mQ = 0)$$

for some elementary maps f_n on A, then all but finitely many f_n are elementary and integrable *on A, and*

$$\lim_{n\to\infty} \int_A f_n$$

exists in E; further, the latter limit does not depend *on the sequence $\{f_n\}$.*

Proof. By Lemma 1,

$$(\forall\, \varepsilon > 0)\ (\exists\, q)\ (\forall\, n, k > q) \quad \int_A |f_n - f| < \varepsilon \text{ and } \int_A |f_n - f_k| < \varepsilon.$$

(The latter *can* be achieved since

$$\lim_{k\to\infty}\int_A |f_n - f_k| = \int_A |f_n - f| < \varepsilon.^3)$$

Now, as

$$|f_n| \le |f_n - f| + |f|,$$

Problem 7 in §5 yields

$$(\forall\, n > k) \quad \int_A |f_n| \le \int_A |f_n - f| + \int_A |f| < \varepsilon + \int_A |f| < \infty.$$

Thus f_n is *elementary and integrable* for $n > k$, as claimed. Also, by Theorem 2 and Corollary 1(ii), both in §4,

$$(\forall\, n, k > q) \quad \left|\int_A f_n - \int_A f_k\right| = \left|\int_A (f_n - f_k)\right| \le \int_A |f_n - f_k| < \varepsilon.$$

Thus $\{\int_A f_n\}$ is a *Cauchy* sequence. As E is complete,

$$\lim \int_A f_n \neq \pm\infty$$

exists in E, as asserted.

Finally, suppose $g_n \to f$ (uniformly) on $A - Q$ for some *other* elementary and integrable maps g_n. By what was shown above, $\lim \int_A g_n$ exists, and

$$\left|\lim \int_A g_n - \lim \int_A f_n\right| = \left|\lim \int_A (g_n - f_n)\right| \le \lim \int_A |g_n - f_n - 0| = 0$$

by Lemma 1, as $g_n - f_n \to 0$ (uniformly) on A. Thus

$$\lim \int_A g_n = \lim \int_A f_n,$$

and all is proved. □

This leads us to the following definition.

***Definition 2.**

If $f\colon S \to E$ is integrable on $A \in \mathcal{M}$ $(mA < \infty)$, we set

$$\int_A f = \int_A f\, dm = \lim_{n\to\infty} \int_A f_n$$

for any elementary and integrable maps f_n such that $f_n \to f$ (uniformly) on $A - Q$, $mQ = 0$.

---

[3] Indeed, $f_n - f_k \to f_n - f$ (*uniformly*) on A as $k \to \infty$; so Lemma 1 applies.

Indeed, such maps *exist* by Theorem 3 of §1, and Lemma 2 excludes ambiguity.

***Note 1.** If f itself is elementary and integrable, Definition 2 agrees with that of §4. For, choosing $f_n = f$ $(n = 1, 2, \ldots)$, we get

$$\int_A f = \int_A f_n$$

(the latter as *in* §4).

***Note 2.** *We may neglect sets on which* $f = 0$, *along with null sets.* For if $f = 0$ on $A - B$ $(A \supseteq B,\ B \in \mathcal{M})$, we may choose $f_n = 0$ on $A - B$ in Definition 2. Then

$$\int_A f = \lim \int_A f_n = \lim \int_B f_n = \int_B f.$$

Thus we now define

$$\int_A f = \int_B f,$$

even if $mA = \infty$, provided $f = 0$ on $A - B$, i.e.,

$$f = f\,C_B \text{ on } A$$

$(C_B =$ characteristic function of $B)$, with $A \supseteq B$, $B \in \mathcal{M}$, and $mB < \infty$.

If such a B exists, we say that f has *m-finite support* in A.

***Note 3.** By Corollary 1 in §5,

$$\int_A |f| < \infty$$

implies that $A(f \neq 0)$ is *σ-finite*. Neglecting $A(f = 0)$, we may assume that

$$A = \bigcup B_n,\ mB_n < \infty, \text{ and } \{B_n\}\uparrow$$

(if not, replace B_n by $\bigcup_{k=1}^{n} B_k$); so $B_n \nearrow A$.

***Lemma 3.** *Let* $\phi\colon S \to E$ *be integrable on* A. *Let* $B_n \nearrow A$, $mB_n < \infty$, *and set*

$$f_n = \phi\, C_{B_n}, \quad n = 1, 2, \ldots.$$

Then $f_n \to \phi$ *(pointwise) on* A, *all* f_n *are integrable on* A, *and*

$$\lim_{n\to\infty} \int_A f_n$$

exists in E. *Furthermore, this limit does not depend on the choice of* $\{B_n\}$.

Proof. Fix any $x \in A$. As $B_n \nearrow A = \bigcup B_n$,

$$(\exists\, n_0)\ (\forall\, n > n_0) \quad x \in B_n.$$

By assumption, $f_n = \phi$ on B_n. Thus

$$(\forall\, n > n_0) \quad f_n(x) = \phi(x);$$

so $f_n \to \phi$ (*pointwise*) on A.

Moreover, $f_n = \phi C_{B_n}$ is m-measurable on A (as ϕ and C_{B_n} are); and

$$|f_n| = |\phi|\, C_{B_n}$$

implies

$$\int_A |f_n| \le \int_A |\phi| < \infty.$$

Thus all f_n are integrable on A.

As $f_n = 0$ on $A - B_n$ $(mB < \infty)$,

$$\int_A f_n$$

is *defined*. Since $f_n \to \phi$ (pointwise) and $|f_n| \le |\phi|$ on A, Theorem 5 in §6, with $g = |\phi|$, yields

$$\int_A |f_n - \phi| \to 0.$$

The rest is as in Lemma 2, with our present Theorem 2 below (assuming m-finite support of f and g), replacing Theorem 2 of §4. Thus all is proved. $\square$

***Definition 3.**

If $\phi\colon S \to E$ is integrable on $A \in \mathcal{M}$, we set

$$\int_A \phi = \int_A \phi\, dm = \lim_{n\to\infty} \int_A f_n,$$

with the f_n as in Lemma 3 (even if ϕ has no m-finite support).

Theorem 2 (linearity)**.** *If $f, g\colon S \to E$ are integrable on $A \in \mathcal{M}$, so is*

$$pf + qg$$

for any scalars p, q. Moreover,

$$\int_A (pf + qg) = p \int_A f + q \int_A g.$$

Furthermore if f and g are scalar valued, p and q may be vectors *in E.*

***Proof.** For the moment, f, g denotes mappings with m-finite support in A.

Integrability is clear since $pf + qg$ is measurable on A (as f and g are), and

$$|pf + qg| \le |p|\,|f| + |q|\,|g|$$

yields

$$\int_A |pf+qg| \le |g| \int_A |f| + |q| \int_A |g| < \infty.$$

Now, as noted above, assume that

$$f = f\,C_{B_1} \text{ and } g = g\,C_{B_2}$$

for some $B_1, B_2 \subseteq A$ $(mB_1 + mB_2 < \infty)$. Let $B = B_1 \cup B_2$; so

$$f = g = pf + qg = 0 \text{ on } A - B;$$

additionally,

$$\int_A f = \int_B f,\ \int_A g = \int_B g, \text{ and } \int_A (pf+qg) = \int_B (pf+qg).$$

Also, $mB < \infty$; so by Definition 2,

$$\int_B f = \lim \int_B f_n \text{ and } \int_B g = \lim \int_B g_n$$

for some elementary and integrable maps

$$f_n \to f \text{ (uniformly) and } g_n \to g \text{ (uniformly) on } B - Q,\ mQ = 0.$$

Thus

$$pf_n + qg_n \to pf + qg \text{ (uniformly) on } B - Q.$$

But by Theorem 2 and Corollary 1(vii), both of §4 (for *elementary and integrable* maps),

$$\int_B (pf_n + qg_n) = p \int_B f_n + q \int_B g_n.$$

Hence

$$\begin{aligned}\int_A (pf+qg) &= \int_B (pf+qg) = \lim \int_B (pf_n + qg_n) \\ &= \lim\left(p \int_B f_n + q \int_B g_n\right) = p \int_B f + q \int_B g = p \int_A f + q \int_A g.\end{aligned}$$

This proves the statement of the theorem, *provided f and g have m-finite support in A*. For the general case, we now resume the notation $f, g, \ldots$ for *any* functions, and extend the result to *any* integrable functions.

Using Definition 3, we set

$$A = \bigcup_{n=1}^{\infty} B_n,\ \{B_n\}\!\uparrow,\ mB_n < \infty,$$

and

$$f_n = f\,C_{B_n},\ g_n = g\,C_{B_n}, \quad n = 1, 2, \ldots.$$

Then by definition,

$$\int_A f = \lim_{n\to\infty} \int_A f_n \text{ and } \int_A g = \lim_{n\to\infty} \int_A g_n,$$

and so

$$p\int_A f + q\int_A g = \lim_{n\to\infty}\left(p\int_A f_n + q\int_A g_n\right).$$

As f_n, g_n have m-finite supports, the first part of the proof yields

$$p\int_A f_n + q\int_A g_n = \int_A (pf_n + qg_n).$$

Thus as claimed,

$$p\int_A f + q\int_A g = \lim \int_A (pf_n + qg_n) = \int_A (pf + qg). \quad \square$$

Similarly, one extends Corollary 1(ii)(iii)(v) of §4 first to maps with m-finite support, and then to *all* integrable maps. The other parts of that corollary need no new proof. (Why?)

Theorem 3 (additivity).

(i) *If $f\colon S \to E$ is integrable on each of n disjoint $\mathcal{M}$-sets A_k, it is so on their union*

$$A = \bigcup_{k=1}^{n} A_k,$$

and

$$\int_A f = \sum_{k=1}^{n} \int_{A_k} f.$$

(ii) *This holds for countable unions, too, if f is integrable on all of A.*

***Proof.** Let f have m-finite support: $f = f\,C_B$ on A, $mB < \infty$. Then

$$\int_A f = \int_B f \text{ and } \int_{A_k} f = \int_{B_k} f,$$

where

$$B_k = A_k \cap B, \quad k = 1, 2, \ldots, n.$$

By Definition 2, fix elementary and integrable maps f_i (on A) and a set Q ($mQ = 0$) such that $f_i \to f$ (uniformly) on $B - Q$ (hence also on $B_k - Q$), with

$$\int_A f = \int_B f = \lim_{i\to\infty} \int_B f_i \text{ and } \int_{A_k} f = \lim_{i\to\infty} \int_{B_k} f_i, \quad k = 1, 2, \ldots, n.$$

As the f_i are *elementary and integrable*, Theorem 1 in §4 yields

$$\int_A f_i = \int_B f_i = \sum_{k=1}^{n} \int_{B_k} f_i = \sum_{k=1}^{n} \int_{A_k} f_i.$$

Hence

$$\int_A f = \lim_{i\to\infty} \int_B f_i = \lim_{i\to\infty} \sum_{k=1}^{n} \int_{B_k} f_i = \sum_{k=1}^{n} \left(\lim_{i\to\infty} \int_{A_k} f_i \right) = \sum_{k=1}^{n} \int_{A_k} f.$$

Thus clause (i) holds for maps *with m-finite support*. For other functions, (i) now follows quite similarly, from Definition 3. (Verify!)

As for (ii), let f be integrable on

$$A = \bigcup_{k=1}^{\infty} A_k \ (\textit{disjoint}), \quad A_k \in \mathcal{M}.$$

In this case, set $g_n = f\,C_{B_n}$, where $B_n = \bigcup_{k=1}^{n} A_k$, $n = 1, 2, \ldots.$ By clause (i), we have

$$\int_A g_n = \int_{B_n} g_n = \sum_{k=1}^{n} \int_{A_k} g_n = \sum_{k=1}^{n} \int_{A_k} f, \tag{3}$$

since $g_n = f$ on each $A_k \subseteq B_n$.

Also, as is easily seen, $|g_n| \le |f|$ on A and $g_n \to f$ (pointwise) on A (proof as in Lemma 3). Thus by Theorem 5 in §6,

$$\int_A |g_n - f| \to 0.$$

As

$$\left| \int_A g_n - \int_A f \right| = \left| \int_A (g_n - f) \right| \le \int_A |g_n - f|,$$

we obtain

$$\int_A f = \lim_{n\to\infty} \int_A g_n,$$

and the result follows by (3). □

Problems on Integration of Complex and Vector-Valued Functions

1. Prove Corollary 1(iii)–(vii) in §4 componentwise for integrable maps $f\colon S \to E^n\ (C^n)$.

2. Prove Theorems 2 and 3 componentwise for $E = E^n\ (C^n)$.

2′. Do it for Corollary 3 in §6.

3. Prove Theorem 1 with

$$\int_A |f| < \infty$$

replaced by

$$\int_A |f_k| < \infty, \quad k = 1, \ldots, n.$$

4. Prove that if $f\colon S \to E^n$ (C^n) is integrable on A, so is $|f|$. Disprove the converse.

5. Disprove Lemma 1 for $mA = \infty$.

***6.** Complete the proof of Lemma 3.

***7.** Complete the proof of Theorem 3.

***8.** Do Problem 1 and $2'$ for $f\colon S \to E$.

***9.** Prove formula (1) from definitions of *Part* II of this section.

⇒**10.** Show that

$$\left| \int_A f \right| \le \int_A |f|$$

for integrable maps $f\colon S \to E$. See also Problem 14.

[Hint: If $mA < \infty$, use Corollary 1(ii) of §4 and Lemma 1. If $mA = \infty$, "imitate" the proof of Lemma 3.]

11. Do Problem 11 in §6 for $f_n\colon S \to E$. Do it componentwise for $E = E^n$ (C^n).

12. Show that if $f, g\colon S \to E^1$ (C) are integrable on A, then[4]

$$\left| \int_A fg \right|^2 \le \int_A |f|^2 \cdot \int_A |g|^2.$$

In what case does *equality* hold? Deduce Theorem 4(c′) in Chapter 3, §§1–3, from this result.

[Hint: Argue as in that theorem. Consider the case

$$(\exists t \in E^1) \quad \int_A |f - tg| = 0.]$$

13. Show that if $f\colon S \to E^1$ (C) is integrable on A and

$$\left| \int_A f \right| = \int_A |f|,$$

then

$$(\exists c \in C) \quad cf = |f| \quad \text{a.e. on } A.$$

---

[4] One may assume that $\int_A |f|^2$ and $\int_A |g|^2$ are *finite* (otherwise, all is trivial).

[Hint: Let $a = \int_A f$. The case $a = 0$ is trivial. If $a \neq 0$, let

$$c = \frac{|a|}{a};\ |c| = 1;\ ca = |a|.$$

Let $r = (cf)_{\text{re}}$. Show that $r \leq |cf| = |f|$,

$$\left|\int_A f\right| = \int_A cf = \int_A r \leq \int_A |f| = \left|\int_A f\right|,$$
$$\int_A |f| = \int_A r = \int_A (cf)_{\text{re}},$$

$(cf)_{\text{re}} = |cf|$ (a.e.), and $cf = |cf| = |f|$ a.e. on A.]

14. Do Problem 10 for $E = C$ using the method of Problem 13.

15. Show that if $f\colon S \to E$ is integrable on A, it is integrable on each $\mathcal{M}$-set $B \subseteq A$. If, in addition,

$$\int_B f = 0$$

for all such B, show that $f = 0$ a.e. on A. Prove it for $E = E^n$ first.
[Hint for $E = E^*$: $A = A(f > 0) \cup A(f \leq 0)$. Use Theorems 1(h) and 2 from §5.]

16. In Problem 15, show that

$$s = \int f$$

is a σ-additive set function on

$$\mathcal{M}_A = \{X \in \mathcal{M} \mid X \subseteq A\}$$

(Note 4 in §5); s is called the *indefinite integral* of f in A.

§8. Product Measures. Iterated Integrals

Let $(X, \mathcal{M}, m)$ and $(Y, \mathcal{N}, n)$ be measure spaces, with $X \in \mathcal{M}$ and $Y \in \mathcal{N}$. Let $\mathcal{C}$ be the family of all "*rectangles*," i.e., sets

$$A \times B,$$

with $A \in \mathcal{M}$, $B \in \mathcal{N}$, $mA < \infty$, and $nB < \infty$.

Define a premeasure $s\colon \mathcal{C} \to E^1$ by

$$s(A \times B) = mA \cdot nB, \quad A \times B \in \mathcal{C}.$$

Let p^* be the s-induced outer measure in $X \times Y$ and

$$p\colon \mathcal{P}^* \to E^*$$

the p^*-induced measure ("*product measure*," $p = m \times n$) on the σ-field $\mathcal{P}^*$ of all p^*-measurable sets in $X \times Y$ (Chapter 6, §§5–6).

We consider functions $f\colon X \times Y \to E^*$ (extended-real).

I. We begin with some definitions.

Definitions.

(1) Given a function $f\colon X \to Y \to E^*$ (of *two* variables x, y), let f_x or $f(x, \cdot\,)$ denote the function *on* Y given by

$$f_x(y) = f(x, y);$$

it arises from f by *fixing* x.

Similarly, f^y or $f(\cdot, y)$ is given by $f^y(x) = f(x, y)$.

(2) Define $g\colon X \to E^*$ by

$$g(x) = \int_Y f(x, \cdot\,)\, dn,$$

and set

$$\int_X \int_Y f\, dn\, dm = \int_X g\, dm,$$

also written

$$\int_X dm(x) \int_Y f(x, y)\, dn(y).$$

This is called the *iterated* integral of f on Y and X, *in this order*.

Similarly,

$$h(y) = \int_X f^y\, dm$$

and

$$\int_Y \int_X f\, dm\, dn = \int_Y h\, dn.$$

Note that by the rules of §5, these integrals are *always* defined.

(3) With f, g, h as above, we say that f is a *Fubini map* or *has the Fubini properties* (after the mathematician Fubini) iff

(a) g is m-measurable on X and h is n-measurable on Y;

(b) f_x is n-measurable on Y *for almost all* x (i.e., for $x \in X - Q$, $mQ = 0$); f^y is m-measurable on X for $y \in Y - Q'$, $nQ' = 0$; and

(c) the iterated integrals above satisfy

$$\int_X \int_Y f\, dn\, dm = \int_Y \int_X f\, dm\, dn = \int_{X\times Y} f\, dp$$

(the main point).

For *monotone sequences*

$$f_k\colon X\times Y\to E^* \quad (k=1,2,\dots),$$

we now obtain the following lemma.

Lemma 1. *If $0\le f_k\nearrow f$ (pointwise) on $X\times Y$ and if each f_k has Fubini property* (a), (b), *or* (c), *then f has the same property.*

Proof. For $k=1,2,\dots,$ set

$$g_k(x)=\int_Y f_k(x,\,\cdot\,)\,dn$$

and

$$h_k(y)=\int_X f_k(\,\cdot\,,y)\,dm.$$

By assumption,

$$0\le f_k(x,\,\cdot\,)\nearrow f(x,\,\cdot\,)$$

pointwise on Y. Thus by Theorem 4 in §6,

$$\int_Y f_k(x,\,\cdot\,)\nearrow\int_Y f(x,\,\cdot\,)\,dn,$$

i.e., $g_k\nearrow g$ (pointwise) on X, with g as in Definition 2.

Again, by Theorem 4 of §6,

$$\int_X g_k\,dm\nearrow\int_X g\,dm;$$

or by Definition 2,

$$\int_X\int_Y f\,dn\,dm=\lim_{k\to\infty}\int_X\int_Y f_k\,dn\,dm.$$

Similarly for

$$\int_Y\int_X f\,dm\,dn$$

and

$$\int_{X\times Y} f\,dp.$$

Hence f satisfies (c) if all f_k do.

Next, let f_k have property (b); so $(\forall\,k)$ $f_k(x,\,\cdot\,)$ is n-measurable on Y if $x\in X-Q_k$ $(mQ_k=0)$. Let

$$Q=\bigcup_{k=1}^{\infty}Q_k;$$

so $mQ = 0$, and *all* $f_k(x, \cdot)$ are n-measurable on Y, for $x \in X - Q$. Hence so is

$$f(x, \cdot) = \lim_{k\to\infty} f_k(x, \cdot).$$

Similarly for $f(\cdot, y)$. Thus f satisfies (b).

Property (a) follows from $g_k \to g$ and $h_k \to h$. $\square$

Using Problems 9 and 10 from §6, the reader will also easily verify the following lemma.

Lemma 2.

(i) *If f_1 and f_2 are nonnegative, p-measurable Fubini maps, so is $af_1 + bf_2$ for $a, b \geq 0$.*

(ii) *If, in addition,*

$$\int_{X\times Y} f_1\, dp < \infty \text{ or } \int_{X\times Y} f_2\, dp < \infty,$$

then $f_1 - f_2$ is a Fubini map, too.

Lemma 3. *Let $f = \sum_{i=1}^{\infty} f_i$ (pointwise), with $f_i \geq 0$ on $X \times Y$.*

(i) *If all f_i are p-measurable Fubini maps, so is f.*

(ii) *If the f_i have Fubini properties* (a) *and* (b), *then*

$$\int_X \int_Y f\, dn\, dm = \sum_{i=1}^{\infty} \int_X \int_Y f_i\, dn\, dm$$

and

$$\int_Y \int_X f\, dm\, dn = \sum_{i=1}^{\infty} \int_Y \int_X f_i\, dm\, dn.$$

II. By Theorem 4 of Chapter 7, §3, the family $\mathcal{C}$ (see above) is a *semiring*, being the product of two *rings*,

$$\{A \in \mathcal{M} \mid mA < \infty\} \text{ and } \{B \in \mathcal{N} \mid nB < \infty\}.$$

(Verify!) Thus using Theorem 2 in Chapter 7, §6, we now show that p is an *extension* of $s\colon \mathcal{C} \to E^1$.

Theorem 1. *The product premeasure s is σ-additive on the semiring $\mathcal{C}$. Hence*

(i) *$\mathcal{C} \subseteq \mathcal{P}^*$ and $p = s < \infty$ on $\mathcal{C}$, and*

(ii) *the characteristic function C_D of any set $D \in \mathcal{C}$ is a Fubini map.*

Proof. Let $D = A \times B \in \mathcal{C}$; so

$$C_D(x, y) = C_A(x) \cdot C_B(y).$$

(Why?) Thus for a *fixed* x, $C_D(x, \cdot\,)$ is just a multiple of the $\mathcal{N}$-simple map C_B, hence n-measurable on Y. Also,

$$g(x) = \int_Y C_D(x, \cdot\,)\, dn = C_A(x) \cdot \int_Y C_B\, dn = C_A(x) \cdot nB;$$

so $g = C_A \cdot nB$ is $\mathcal{M}$-simple on X, with

$$\int_X \int_Y C_D\, dn\, dm = \int_X g\, dm = nB \int_X C_A\, dm = nB \cdot mA = sD.$$

Similarly for $C_D(\,\cdot\,, y)$, and

$$h(y) = \int_X C_D(\,\cdot\,, y)\, dm.$$

Thus C_D has Fubini properties (a) and (b), and for every $D \in \mathcal{C}$

$$\int_X \int_Y C_D\, dn\, dm = \int_Y \int_X C_D\, dm\, dn = sD. \tag{1}$$

To prove σ-additivity, let

$$D = \bigcup_{i=1}^{\infty} D_i \ (disjoint),\ D_i \in \mathcal{C};$$

so

$$C_D = \sum_{i=1}^{\infty} C_{D_i}.$$

(Why?) As shown above, each C_{D_i} has Fubini properties (a) and (b); so by (1) and Lemma 3,

$$sD = \int_X \int_Y C_D\, dn\, dm = \sum_{i=1}^{\infty} \int_X \int_Y C_{D_i}\, dn\, dm = \sum_{i=1}^{\infty} sD_i,$$

as required.

Assertion (i) now follows by Theorem 2 in Chapter 7, §6. Hence

$$sD = pD = \int_{X \times Y} C_D\, dp;$$

so by formula (1), C_D also has Fubini property (c), and all is proved. □

Next, let $\mathcal{P}$ be the *σ-ring generated by the semiring* $\mathcal{C}$ (so $\mathcal{C} \subseteq \mathcal{P} \subseteq \mathcal{P}^*$).

Lemma 4. *$\mathcal{P}$ is the* least *set family $\mathcal{R}$ such that*

(i) $\mathcal{R} \supseteq \mathcal{C}$;

(ii) *$\mathcal{R}$ is closed under countable disjoint unions; and*

(iii) *$H - D \in \mathcal{R}$ if $D \in \mathcal{R}$ and $D \subseteq H$, $H \in \mathcal{C}$.*

This is simply Theorem 3 in Chapter 7, §3, with changed notation.

Lemma 5. *If $D \in \mathcal{P}$ (σ-generated by $\mathcal{C}$), then C_D is a Fubini map.*

Proof. Let $\mathcal{R}$ be the family of all $D \in \mathcal{P}$ such that C_D is a Fubini map. We shall show that $\mathcal{R}$ satisfies (i)–(iii) of Lemma 4, and so $\mathcal{P} \subseteq \mathcal{R}$.

(i) By Theorem 1, each C_D ($D \in \mathcal{C}$) is a Fubini map; so each $D \in \mathcal{C}$ is in $\mathcal{R}$.

(ii) Let

$$D = \bigcup_{i=1}^{\infty} D_i \ (\textit{disjoint}), \quad D_i \in \mathcal{R}.$$

Then

$$C_D = \sum_{i=1}^{\infty} C_{D_i},$$

and each C_{D_i} is a Fubini map. Hence so is C_D by Lemma 3. Thus $D \in \mathcal{R}$, proving (ii).

(iii) We must show that C_{H-D} is a Fubini map if C_D is and if $D \subseteq H$, $H \in \mathcal{C}$. Now, $D \subseteq H$ implies

$$C_{H-D} = C_H - C_D.$$

(Why?) Also, by Theorem 1, $H \in \mathcal{C}$ implies

$$\int_{X \times Y} C_H \, dp = pH = sH < \infty,$$

and C_H is a Fubini map. So is C_D by assumption. So also is

$$C_{H-D} = C_H - C_D$$

by Lemma 2(ii). Thus $H - D \in \mathcal{R}$, proving (iii).

By Lemma 4, then, $\mathcal{P} \subseteq \mathcal{R}$. Hence $(\forall\, D \in \mathcal{P})$ C_D is a Fubini map. □

We can now establish one of the main theorems, due to Fubini.

Theorem 2 (Fubini I). *Suppose $f \colon X \times Y \to E^*$ is $\mathcal{P}$-measurable on $X \times Y$ ($\mathcal{P}$ as above) rom. Then f is a Fubini map if either*

(i) *$f \geq 0$ on $X \times Y$, or*

(ii) one of

$$\int_{X\times Y} |f|\,dp,\ \int_X \int_Y |f|\,dn\,dm,\ \text{ or } \int_Y \int_X |f|\,dm\,dn$$

is finite.[1]

In both cases,

$$\int_X \int_Y f\,dn\,dm = \int_Y \int_X f\,dm\,dn = \int_{X\times Y} f\,dp. \tag{2}$$

Proof. First, let

$$f = \sum_{i=1}^{\infty} a_i C_{D_i} \quad (a_i \ge 0,\ D_i \in \mathcal{P}),$$

i.e., f is $\mathcal{P}$-*elementary*, hence certainly p-measurable. (Why?) By Lemmas 5 and 2, each $a_i C_{D_i}$ is a Fubini map. Hence so is f (Lemma 3). Formula (2) is simply Fubini property (c).

Now take any $\mathcal{P}$-*measurable* $f \ge 0$. By Lemma 2 in §2,

$$f = \lim_{k\to\infty} f_k \text{ on } X \times Y$$

for some sequence $\{f_k\}\uparrow$ of $\mathcal{P}$-elementary maps, $f_k \ge 0$. As shown above, each f_k is a Fubini map. Hence so is f by Lemma 1. This settles case (i).

Next, assume (ii). As f is $\mathcal{P}$-measurable, so are f^+, f_-, and $|f|$ (Theorem 2 in §2). As they are nonnegative, they are Fubini maps by case (i).

So is $f = f^+ - f^-$ by Lemma 2(ii), since $f^+ \le |f|$ implies

$$\int_{X\times Y} f^+\,dp < \infty$$

by our assumption (ii). (The three integrals are *equal*, as $|f|$ is a Fubini map.)

Thus all is proved. □

III. We now want to replace $\mathcal{P}$ by $\mathcal{P}^*$ in Lemma 5 and Theorem 2. This works only under certain σ-finiteness conditions, as shown below.

Lemma 6. *Let* $D \in \mathcal{P}^*$ *be* σ-finite, *i.e.,*

$$D = \bigcup_{i=1}^{\infty} D_i \ (\textit{disjoint})$$

for some $D_i \in \mathcal{P}^*$, *with* $pD_i < \infty$ $(i = 1, 2, \ldots)$.[2]

---

[1] Note the use of *absolute values*; without them, Theorem 2 fails (see Problem 5′).

[2] See Note 2 in Chapter 7, §8.

Then there is a $K \in \mathcal{P}$ *such that* $p(K - D) = 0$ *and* $D \subseteq K$*.*

Proof. As $\mathcal{P}$ is a σ-ring containing $\mathcal{C}$, it also contains $\mathcal{C}_\sigma$. Thus by Theorem 3 of Chapter 7, §5, p^* is $\mathcal{P}$*-regular*.

For the rest, proceed as in Theorems 1 and 2 in Chapter 7, §7. □

Lemma 7. *If* $D \in \mathcal{P}^*$ *is* σ*-finite* (*Lemma* 6), *then* C_D *is a Fubini map.*

Proof. By Lemma 6,

$$(\exists K \in \mathcal{P}) \quad p(K - D) = 0, \ D \subseteq K.$$

Let $Q = K - D$, so $pQ = 0$, and $C_Q = C_K - C_D$; that is, $C_D = C_K - C_Q$ and

$$\int_{X \times Y} C_Q \, dp = pQ = 0.$$

As $K \in \mathcal{P}$, C_K is a Fubini map. Thus by Lemma 2(ii), all reduces to proving the same for C_Q.

Now, as $pQ = 0$, Q is certainly σ-finite; so by Lemma 6,

$$(\exists Z \in \mathcal{P}) \quad Q \subseteq Z, \ pZ = pQ = 0.$$

Again C_Z is a Fubini map; so

$$\int_X \int_Y C_Z \, dn \, dm = \int_{X \times Y} C_Z \, dp = pZ = 0.$$

As $Q \subseteq Z$, we have $C_Q \le C_Z$, and so

$$\begin{aligned} \int_X \int_Y C_Q \, dn \, dm &= \int_X \left[\int_Y C_Q(x, \cdot\,) \, dn \right] dm \\ &\le \int_X \left[\int_Y C_Z(x, \cdot\,) \, dn \right] dm = \int_{X \times Y} C_Z \, dp = 0. \end{aligned} \tag{3}$$

Similarly,

$$\int_Y \int_X C_Q \, dm \, dn = \int_Y \left[\int_X C_Q(\,\cdot\,, y) \, dm \right] dn = 0. \tag{4}$$

Thus setting

$$g(x) = \int_Y C_Q(x, \cdot\,) \, dn \text{ and } h(y) = \int_X C_Q(\,\cdot\,, y) \, dm,$$

we have

$$\int_X g \, dm = 0 = \int_Y h \, dn.$$

Hence by Theorem 1(h) in §5, $g = 0$ a.e. on X, and $h = 0$ a.e. on Y. So g and h are "almost" measurable (Definition 2 of §3); i.e., C_Q *has Fubini property* (a).

Similarly, one establishes (b), and (3) yields Fubini property (c), since

$$\int_X \int_Y C_Q \, dn \, dm = \int_Y \int_X C_Q \, dm \, dn = \int_{X\times Y} C_Q \, dp = 0,$$

as required. □

Theorem 3 (Fubini II). *Suppose $f\colon X\times Y \to E^*$ is $\mathcal{P}^*$-measurable*[3] *on $X\times Y$ and satisfies condition* (i) *or* (ii) *of Theorem* 2.

*Then f is a Fubini map, provided f has σ-*finite support, *i.e., f vanishes outside some σ-finite set $H \subseteq X \times Y$.*

Proof. First, let

$$f = \sum_{i=1}^{\infty} a_i C_{D_i} \quad (a_i > 0,\ D_i \in \mathcal{P}^*),$$

with $f = 0$ on $-H$ (as above).

As $f = a_i \neq 0$ on A_i, we must have $D_i \subseteq H$; so all D_i are σ-finite. (Why?) Thus by Lemma 7, each C_{D_i} is a Fubini map, and so is f. (Why?)

If f is $\mathcal{P}^*$-measurable and nonnegative, and $f = 0$ on $-H$, we can proceed as in Theorem 2, *making all f_k vanish on $-H$.* Then the f_k and f are Fubini maps by what was shown above.

Finally, in case (ii), $f = 0$ on $-H$ implies

$$f^+ = f^- = |f| = 0 \text{ on } -H.$$

Thus f^+, f^-, and f are Fubini maps by part (i) and the argument of Theorem 2. □

Note 1. The σ-finite support is automatic if f is *p-integrable* (Corollary 1 in §5), or if p or both m and n are σ-finite (see Problem 3). The condition is also redundant if f is *$\mathcal{P}$-measurable* (Theorem 2; see also Problem 4).

Note 2. By induction, our definitions and Theorems 2 and 3 extend to any finite number q of measure spaces

$$(X_i, \mathcal{M}_i, m_i), \quad i = 1, \ldots, q.$$

One writes

$$p = m_1 \times m_2$$

if $q = 2$ and sets

$$m_1 \times m_2 \times \cdots \times m_{q+1} = (m_1 \times \cdots \times m_q) \times m_{q+1}.$$

---

[3] Or, equivalently, *p-measurable* (Note 2 in §3), as p is *complete* (Theorem 1 of Chapter 7, §6).

Theorems 2 and 3 with similar assumptions then state that the *order* of integrations is immaterial.

Note 3. Lebesgue measure in E^q can be treated as the product of q one-dimensional measures. Similarly for *LS product measures* (but this method is less general than that described in Problems 9 and 10 of Chapter 7, §9).

IV. Theorems 2(ii) and 3(ii) hold also for functions

$$f\colon X \times Y \to E^n\,(C^n)$$

if Definitions 2 and 3 are modified as follows (so that they make sense for such maps): In Definition 2, set

$$g(x) = \int_Y f_x\,dn$$

if f_x is *n-integrable* on Y, and $g(x) = 0$ otherwise. Similarly for $h(y)$. In Definition 3, replace "measurable" by "integrable."

For the proof of the theorems, apply Theorems 2(i) and 3(i) *to* $|f|$. This yields

$$\int_Y \int_X |f|\,dm\,dn = \int_X \int_Y |f|\,dn\,dm = \int_{X\times Y} |f|\,dp.$$

Hence if one of these integrals is finite, f is p-integrable on $X \times Y$, and so are its q components. The result then follows on noting that f is a Fubini map (in the modified sense) iff its components are. (Verify!) See also Problem 12 below.

V. In conclusion, note that integrals of the form

$$\int_D f\,dp \quad (D \in \mathcal{P}^*)$$

reduce to

$$\int_{X\times Y} f \cdot C_D\,dp.$$

Thus it suffices to consider integrals *over* $X \times Y$.

Problems on Product Measures and Fubini Theorems

1. Prove Lemmas 2 and 3.

1′. Show that $\{A \in \mathcal{M} \mid mA < \infty\}$ is a set *ring*.

2. Fill in all proof details in Theorems 1 to 3.

2′. Do the same for Lemmas 5 to 7.

3. Prove that if m and n are σ-finite, so is $p = m \times n$. Disprove the converse by an example.

[Hint: $(\bigcup_i A_i) \times (\bigcup_j B_j) = \bigcup_{i,j} (A_i \times B_j)$. Verify!]

4. Prove the following.

(i) Each $D \in \mathcal{P}$ (as in the text) is (p) σ-finite.

(ii) All $\mathcal{P}$-measurable maps $f\colon X \times Y \to E^*$ have σ-finite support.

[Hints: (i) Use Problem 14(b) from Chapter 7, §3. (ii) Use (i) for $\mathcal{P}$-elementary and nonnegative maps first.]

5. (i) Find $D \in \mathcal{P}^*$ and $x \in X$ such that $C_D(x, \cdot\,)$ is *not* n-measurable on Y. Does this contradict Lemma 7?

[Hint: Let $m = n =$ Lebesgue measure in E^1; $D = \{x\} \times Q$, with Q *nonmeasurable*.]

(ii) Which $\mathcal{C}$-sets have nonzero measure if $X = Y = E^1$, m^* is as in Problem 2(b) of Chapter 7, §5 (with $S = X$), and n is Lebesgue measure?

5′. Let $m = n =$ Lebesgue measure in $[0, 1] = X = Y$. Let

$$f_k = \begin{cases} k(k+1) & \text{on } \left(\dfrac{1}{k+1}, \dfrac{1}{k}\right] \text{ and} \\ 0 & \text{elsewhere.} \end{cases}$$

Let

$$f(x, y) = \sum_{k=1}^{\infty} [f_k(x) - f_{k+1}(x)]\, f_k(y);$$

the series *converges*. (Why?) Show that

(i) $(\forall\, k)\ \int_X f_k = 1$;

(ii) $\int_X \int_Y f\, dn\, dm = 1 \neq 0 = \int_Y \int_X f\, dm\, dn$.

What is wrong? Is f $\mathcal{P}$-measurable?

[Hint: Explore

$$\int_X \int_Y |f|\, dn\, dm.]$$

6. Let $X = Y = [0, 1]$, m as in Example (c) of Chapter 7, §6, $(S = X)$ and $n =$ Lebesgue measure in Y.

(i) Show that $p = m \times n$ is a topological measure under the standard metric in E^2.

(ii) Prove that $D = \{(x, y) \in X \times Y \mid x = y\} \in \mathcal{P}^*$.

(iii) Describe $\mathcal{C}$.

[Hints: (i) Any *subinterval* of $X \times Y$ is in $\mathcal{P}^*$; (ii) D is *closed*. Verify!]

7. Continuing Problem 6, let $f = C_D$.

(i) Show that

$$\int_Y \int_X f\,dn\,dm = 0 \neq 1 = \int_Y \int_X f\,dm\,dn.$$

What is wrong?

[Hint: D is not σ-finite; for if

$$D = \bigcup_{i=1}^{\infty} D_i,$$

at least one D_i is uncountable and has no *finite* basic covering values (why?), so $p^* D_i = \infty$.]

(ii) Compute $p^*\{(x, 0) \mid x \in X\}$ and $p^*\{(0, y) \mid y \in Y\}$.

8. Show that $D \in \mathcal{P}^*$ is σ-finite iff

$$D \subseteq \bigcup_{i=1}^{\infty} D_i \ (\textit{disjoint})$$

for some sets $D_i \in \mathcal{C}$.

[Hint: First let $p^* D < \infty$. Use Corollary 1 from Chapter 7, §1.]

9. Given $D \in \mathcal{P}$, $a \in X$, and $b \in Y$, let

$$D_a = \{y \in Y \mid (a, y) \in D\}$$

and

$$D^b = \{x \in X \mid (x, b) \in D\}.$$

(See Figure 34 for $X = Y = E^1$.)

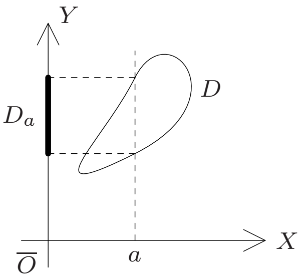

FIGURE 34

Prove that

(i) $D_a \in \mathcal{N}$, $D^b \in \mathcal{M}$;

(ii) $C_D(a, \cdot\,) = C_{D_a}$, $nD_a = \int_Y C_D(a, \cdot\,)\,dn$, $mD^b = \int_X C_D(\,\cdot\,, b)\,dm$.

[Hint: Let

$$\mathcal{R} = \{Z \in \mathcal{P} \mid Z_a \in \mathcal{N}\}.$$

Show that $\mathcal{R}$ is a σ-ring $\supseteq \mathcal{C}$. Hence $\mathcal{R} \supseteq \mathcal{P}$; $D \in \mathcal{R}$; $D_a \in \mathcal{N}$. Similarly for D^b.]

⇒**10.** Let $m = n =$ Lebesgue measure in $E^1 = X = Y$. Let $f\colon E^1 \to [0, \infty)$ be m-measurable on X. Let

$$H = \{(x, y) \in E^2 \mid 0 \leq y < f(x)\}$$

and

$$G = \{(x, y) \in E^2 \mid y = f(x, y)\}$$

(the "graph" of f). Prove that

(i) $H \in \mathcal{P}^*$ and

$$pH = \int_X f\,dm$$

(= "the area under f");

(ii) $G \in \mathcal{P}^*$ and $pG = 0$.

[Hints: (i) First take $f = C_D$, and elementary and nonnegative maps. Then use Lemma 2 in §2 (last clause). Fix elementary and nonnegative maps $f_k \nearrow f$, assuming $f_k < f$ (if not, replace f_k by $(1 - \frac{1}{k})f_k$). Let

$$H_k = \{(x, y) \mid 0 \leq y < f_k(x)\}.$$

Show that $H_k \nearrow H \in \mathcal{P}^*$.

(ii) Set

$$\phi(x, y) = y - f(x).$$

Using Corollary 4 of §1, show that ϕ is p-measurable on E^2; so $G = E^2(\phi = 0) \in \mathcal{P}^*$. Dropping a null set (Lemma 6), assume $G \in \mathcal{P}$. By Problem 9(ii),

$$(\forall\, x \in E^1) \quad \int_Y C_G(x, \cdot\,)\, dn = nG_x = 0,$$

as $G_x = \{f(x)\}$, a *singleton*.]

11. Let

$$f(x, y) = \phi_1(x)\phi_2(y).$$

Prove that if ϕ_1 is m-integrable on X and ϕ_2 is n-integrable on Y, then f is p-integrable on $X \times Y$ and

$$\int_{X\times Y} f\,dp = \int_X \phi_1 \cdot \int_Y \phi_2.$$

***12.** Prove Theorem 3(ii) for $f\colon X \times Y \to E$ (E complete).

[Outline: If f is $\mathcal{P}^*$-*simple*, use Lemma 7 above and Theorem 2 in §7.

If

$$f = \sum_{k=1}^{\infty} a_k\, C_{D_k}, \quad D_k \in \mathcal{P}^*,$$

let

$$H_k = \bigcup_{i=1}^{k} D_i$$

and $f_k = fC_{H_k}$, so the f_k are $\mathcal{P}^*$-simple (hence Fubini maps), and $f_k \to f$ (pointwise) on $X \times Y$, with $|f_k| \leq |f|$ and

$$\int_{X\times Y} |f|\, dp < \infty$$

(by assumption). Now use Theorem 5 from §6.

Let now f be $\mathcal{P}^*$-measurable; so

$$f = \lim_{k\to\infty} f_k \text{ (uniformly)}$$

for some $\mathcal{P}^*$-elementary maps g_k (Theorem 3 in §1). By assumption, $f = fC_H$ (H σ-finite); so we may assume $g_k = g_k C_H$. Then as shown above, all g_k are Fubini maps. So is f by Lemma 1 in §7 (verify!), *provided* $H \subseteq D$ *for some* $D \in \mathcal{C}$.

In the general case, by Problem 8,

$$H \subseteq \bigcup_i D_i \ (disjoint),\ D_i \in \mathcal{C}.$$

Let $H_i = H \cap D_i$. By the previous step, each fC_{H_i} is a Fubini map; so is

$$f_k = \sum_{i=1}^{k} fC_{H_i}$$

(why?), hence so is $f = \lim_{k\to\infty} f_k$, by Theorem 5 of §6. (Verify!)]

13. Let m = Lebesgue measure in E^1, p = Lebesgue measure in E^s, $X = (0, \infty)$, and

$$Y = \{\bar{y} \in E^s \mid |\bar{y}| = 1\}.$$

Given $\bar{x} \in E^s - \{\bar{0}\}$, let

$$r = |\bar{x}| \text{ and } \bar{u} = \frac{\bar{x}}{r} \in Y.$$

Call r and $\bar{u}$ the *polar coordinates* of $\bar{x} \neq \bar{0}$.

If $D \subseteq Y$, set

$$n^*D = s \cdot p^*\{r\bar{u} \mid \bar{u} \in D,\ 0 < r \le 1\}.$$

Show that n^* is an outer measure in Y; so it induces a measure n in Y. Then prove that

$$\int_{E^s} f\,dp = \int_X r^{s-1}\,dm(r) \int_Y f(r\bar{u})\,dn(\bar{u})$$

if f is p-measurable and nonnegative on E^s.

[Hint: Start with $f = C_A$,

$$A = \{r\bar{u} \mid \bar{u} \in H,\ a < r < b\},$$

for some *open* set $H \subseteq Y$ (subspace of E^s). Next, let $A \in \mathcal{B}$ (Borel set in Y); then $A \subseteq \mathcal{P}^*$. Then let f be p-elementary, and so on.]

§9. Riemann Integration. Stieltjes Integrals

I. In this section, $\mathcal{C}$ is the family of all intervals in E^n, and m is an additive finite premeasure on $\mathcal{C}$ (or $\mathcal{C}_s$), such as the *volume function* v (Chapter 7, §§1–2).

By a $\mathcal{C}$*-partition* of $A \in \mathcal{C}$ (or $A \in \mathcal{C}_s$), we mean a *finite* family

$$\mathcal{P} = \{A_i\} \subset \mathcal{C}$$

such that

$$A = \bigcup_i A_i \ (\textit{disjoint}).$$

As we noted in §5, the *Riemann integral*,

$$R\int_A f = R\int_A f\,dm,$$

of $f\colon E^n \to E^1$ can be defined as its Lebesgue counterpart,

$$\int_A f,$$

with elementary maps replaced by *simple step functions* ("$\mathcal{C}$*-simple*" maps.) Equivalently, one can use the following construction, due to J. G. Darboux.

Definitions.

(a) Given $f\colon E^n \to E^*$ and a $\mathcal{C}$-partition

$$\mathcal{P} = \{A_1, \ldots, A_q\}$$

of A, we define the *lower* and *upper* Darboux sums, $\underline{S}$ and $\overline{S}$, of f over $\mathcal{P}$ (with respect to m) by

$$\underline{S}(f,\mathcal{P}) = \sum_{i=1}^{q} mA_i \cdot \inf f[A_i] \text{ and } \overline{S}(f,\mathcal{P}) = \sum_{i=1}^{q} mA_i \cdot \sup f[A_i].^{1} \tag{1}$$

(b) The *lower* and *upper Riemann integrals* ("*R-integrals*") of f on A (with respect to m) are

$$\left.\begin{aligned} R\underline{\int_A} f = R\underline{\int_A} f\,dm &= \sup_{\mathcal{P}} \underline{S}(f,\mathcal{P}) \text{ and} \\ R\overline{\int_A} f = R\overline{\int_A} f\,dm &= \inf_{\mathcal{P}} \overline{S}(f,\mathcal{P}),^{2} \end{aligned}\right\} \tag{2}$$

where the "inf" and "sup" are taken over all $\mathcal{C}$-partitions $\mathcal{P}$ of A.

(c) We say that f is *Riemann-integrable* ("*R-integrable*") with respect to m on A iff f is *bounded* on A and

$$R\underline{\int_A} f = R\overline{\int_A} f.$$

[1,2] These expressions *exist* in E^* (Chapter 4, §4, (2*)).

We then set

$$R\int_A f = R\underline{\int}_A f = R\overline{\int}_A f\,dm = R\int_A f\,dm$$

and call it the *Riemann integral* ("*R-integral*") of f on A. "Classical" notation:

$$R\int_A f(\bar{x})\,dm(\bar{x}).$$

If $A = [a, b] \subset E^1$, we also write

$$R\int_a^b f = R\int_a^b f(x)\,dm(x)$$

instead.

If m is *Lebesgue* measure (or premeasure) in E^1, we write "dx" for "$dm(x)$."

For *Lebesgue* integrals, we replace "R" by "L," or we simply omit "R."

If f is R-integrable on A, we also say that

$$R\int_A f$$

exists (note that this implies the *boundedness* of f); note that

$$R\underline{\int}_A f \text{ and } R\overline{\int}_A f$$

are *always* defined in E^*.

Below, we always restrict f to a fixed $A \in \mathcal{C}$ (or $A \in \mathcal{C}_s$); $\mathcal{P}$, $\mathcal{P}'$, $\mathcal{P}''$, $\mathcal{P}^*$, and $\mathcal{P}_k$ denote $\mathcal{C}$-partitions of A.

We now obtain the following result for *any* additive $m\colon \mathcal{C} \to [0, \infty)$.

Corollary 1. *If* $\mathcal{P}$ refines $\mathcal{P}'$ (§1), *then*

$$\underline{S}(f, \mathcal{P}') \leq \underline{S}(f, \mathcal{P}) \leq \overline{S}(f, \mathcal{P}) \leq \overline{S}(f, \mathcal{P}').$$

Proof. Let $\mathcal{P}' = \{A_i\}$, $\mathcal{P} = \{B_{ik}\}$, and

$$(\forall\, i) \quad A_i = \bigcup_k B_{ik}.$$

By additivity,

$$mA_i = \sum_k mB_{ik}.$$

Also, $B_{ik} \subseteq A_i$ implies

$$\begin{gathered} f[B_{ik}] \subseteq f[A_i]; \\ \sup f[B_{ik}] \leq \sup f[A_i]; \text{ and} \\ \inf f[B_{ik}] \geq \inf f[A_i]. \end{gathered}$$

So setting

$$a_i = \inf f[A_i] \text{ and } b_{ik} = \inf f[B_{ik}],$$

we get

$$\begin{aligned} \underline{S}(f, \mathcal{P}') = \sum_i a_i m A_i &= \sum_i \sum_k a_i m B_{ik} \\ &\leq \sum_{i,k} b_{ik} m B_{ik} = \underline{S}(f, \mathcal{P}). \end{aligned}$$

Similarly,

$$\overline{S}(f, \mathcal{P}') \leq \overline{S}(f, \mathcal{P}),$$

and

$$\underline{S}(f, \mathcal{P}) \leq \overline{S}(f, \mathcal{P})$$

is obvious from (1). □

Corollary 2. *For any $\mathcal{P}'$ and $\mathcal{P}''$,*

$$\underline{S}(f, \mathcal{P}') \leq \overline{S}(f, \mathcal{P}'').$$

Hence

$$R \underline{\int}_A f \leq R \overline{\int}_A f.$$

Proof. Let $\mathcal{P} = \mathcal{P}' \cap \mathcal{P}''$ (see §1). As $\mathcal{P}$ refines both $\mathcal{P}'$ and $\mathcal{P}''$, Corollary 1 yields

$$\underline{S}(f, \mathcal{P}') \leq \underline{S}(f, \mathcal{P}) \leq \overline{S}(f, \mathcal{P}) \leq \overline{S}(f, \mathcal{P}'').$$

Thus, indeed, no lower sum $\underline{S}(f, \mathcal{P}')$ exceeds any upper sum $\overline{S}(f, \mathcal{P}'')$.

Hence also

$$\sup_{\mathcal{P}'} \underline{S}(f, \mathcal{P}') \leq \inf_{\mathcal{P}''} \overline{S}(f, \mathcal{P}''),$$

i.e.,

$$R \underline{\int}_A f \leq R \overline{\int}_A f,$$

as claimed. □

Lemma 1. *A map $f\colon A \to E^1$ is R-integrable iff f is bounded and, moreover,*

$$(\forall \varepsilon > 0)\ (\exists \mathcal{P}) \quad \overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \varepsilon. \tag{3}$$

Proof. By formulas (1) and (2),

$$\underline{S}(f, \mathcal{P}) \le R\underline{\int_A} f \le R\overline{\int_A} f \le \overline{S}(f, \mathcal{P}).$$

Hence (3) implies

$$\left| R\overline{\int_A} f - R\underline{\int_A} f \right| < \varepsilon.$$

As ε is arbitrary, we get

$$R\overline{\int_A} f = R\underline{\int_A} f;$$

so f is R-integrable.

Conversely, if so, definitions (b) and (c) imply the existence of $\mathcal{P}'$ and $\mathcal{P}''$ such that

$$\underline{S}(f, \mathcal{P}') > R\int_A f - \frac{1}{2}\varepsilon$$

and

$$\overline{S}(f, \mathcal{P}'') < R\int_A f + \frac{1}{2}\varepsilon.$$

Let $\mathcal{P}$ refine both $\mathcal{P}'$ and $\mathcal{P}''$. Then by Corollary 1,

$$\begin{aligned} \overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) &\le \overline{S}(f, \mathcal{P}'') - \underline{S}(f, \mathcal{P}') \\ &< \left(R\int_A f + \frac{1}{2}\varepsilon \right) - \left(R\int_A f - \frac{1}{2}\varepsilon \right) = \varepsilon, \end{aligned}$$

as required. □

Lemma 2. *Let f be $\mathcal{C}$-simple; say, $f = a_i$ on A_i for some $\mathcal{C}$-partition $\mathcal{P}^* = \{A_i\}$ of A (we then write*

$$f = \sum_i a_i C_{A_i}$$

on A; see Note 4 of §4).

Then

$$R\underline{\int_A} f = R\overline{\int_A} f = \underline{S}(f, \mathcal{P}^*) = \overline{S}(f, \mathcal{P}^*) = \sum_i a_i m A_i. \tag{4}$$

Hence any finite $\mathcal{C}$-simple function is R-integrable, with $R\int_A f$ as in (4).

Proof. Given *any* $\mathcal{C}$-partition $\mathcal{P} = \{B_k\}$ of A, consider

$$\mathcal{P}^* \Cap \mathcal{P} = \{A_i \cap B_k\}.$$

As $f = a_i$ on $A_i \cap B_k$ (even on *all* of A_i),

$$a_i = \inf f[A_i \cap B_k] = \sup f[A_i \cap B_k].$$

Also,

$$A = \bigcup_{i,k}(A_i \cap B_k) \ (disjoint)$$

and

$$(\forall\, i) \quad A_i = \bigcup_k (A_i \cap B_k);$$

so

$$mA_i = \sum_k m(A_i \cap B_k)$$

and

$$\underline{S}(f, \mathcal{P}) = \sum_i \sum_k a_i\, m(A_i \cap B_k) = \sum_i a_i\, mA_i = \underline{S}(f, \mathcal{P}^*)$$

for *any* such $\mathcal{P}$.

Hence also

$$\sum_i a_i\, mA_i = \sup_{\mathcal{P}} \underline{S}(f, \mathcal{P}) = R\underline{\int}_A f.$$

Similarly for $R\overline{\int}_A f$. This proves (4).

If, further, f is finite, it is *bounded* (by $\max |a_i|$) since there are only *finitely* many a_i; so f is R-integrable on A, and all is proved. $\square$

Note 1. Thus $\underline{S}$ and $\overline{S}$ are *integrals of* $\mathcal{C}$*-simple maps*, and definition (b) can be restated:

$$R\underline{\int}_A f = \sup_g R\int_A g \text{ and } R\overline{\int}_A f = \inf_h R\int_A h,$$

taking the sup and inf over all $\mathcal{C}$-simple maps g, h with

$$g \le f \le h \text{ on } A.$$

(*Verify* by properties of glb and lub!)

Therefore, we can now develop R-integration as in §§4–5, replacing elementary maps by $\mathcal{C}$-simple maps, with $S = E^n$. In particular, Problem 5 in §5 works out as before.

Hence *linearity* (Theorem 1 of §6) follows, with the same proof. One also obtains *additivity* (limited to $\mathcal{C}$-partitions). Moreover, the R-integrability of f and g implies that of fg, $f \vee g$, $f \wedge g$, and $|f|$. (See the Problems.)

Theorem 1. *If $f_i \to f$ (uniformly) on A and if the f_i are R-integrable on A, so also is f. Moreover,*

$$\lim_{i\to\infty} R\int_A |f-f_i| = 0 \text{ and } \lim_{i\to\infty} R\int_A f_i = R\int_A f.$$

Proof. As all f_i are bounded (definition (c)), so is f, by Problem 10 of Chapter 4, §12.

Now, given $\varepsilon > 0$, fix k such that

$$(\forall\, i \geq k) \quad |f-f_i| < \frac{\varepsilon}{mA} \quad \text{on } A.$$

Verify that

$$(\forall\, i \geq k)\ (\forall\, \mathcal{P}) \quad |\underline{S}(f-f_i,\mathcal{P})| < \varepsilon \text{ and } |\overline{S}(f-f_i,\mathcal{P})| < \varepsilon;$$

fix one such f_i and choose a $\mathcal{P}$ such that

$$\overline{S}(f_i,\mathcal{P}) - \underline{S}(f_i,\mathcal{P}) < \varepsilon,$$

which one can do by Lemma 1. Then for *this* $\mathcal{P}$,

$$\overline{S}(f,\mathcal{P}) - \underline{S}(f,\mathcal{P}) < 3\varepsilon.$$

(Why?) By Lemma 1, then, f is R-integrable on A.

Finally,

$$\begin{aligned}\left|R\int_A f - R\int_A f_i\right| &\leq R\int_A |f-f_i| \\ &\leq R\int_A \Big(\frac{\varepsilon}{mA}\Big) = mA\Big(\frac{\varepsilon}{mA}\Big) = \varepsilon\end{aligned}$$

for all $i \geq k$. Hence the second clause of our theorem follows, too. □

Corollary 3. *If $f\colon E^1 \to E^1$ is bounded and* regulated (*Chapter* 5, §10) *on $A = [a,b]$, then f is R-integrable on A.*

In particular, this applies if f is monotone, or of bounded variation, or relatively continuous, or a step function, on A.

Proof. By Lemma 2, this applies to $\mathcal{C}$-simple maps.

Now, let f be regulated (e.g., of the kind specified above).

Then by Lemma 2 of Chapter 5, §10,

$$f = \lim_{i\to\infty} g_i \quad \text{(uniformly)}$$

for finite $\mathcal{C}$-simple g_i.

Thus f is R-integrable on A by Theorem 1. □

II. Henceforth, we assume that m is a *measure* on a σ-ring $\mathcal{M} \supseteq \mathcal{C}$ in E^n, with $m < \infty$ on $\mathcal{C}$. (For a reader who took the "limited approach," it is now time to consider §§4–6 in full.) The measure m may, but *need not*, be Lebesgue measure in E^n.

Theorem 2. *If $f\colon E^n \to E^1$ is R-integrable on $A \in \mathcal{C}$, it is also Lebesgue integrable (with respect to m as above) on A, and*

$$L\int_A f = R\int_A f.$$

Proof. Given a $\mathcal{C}$-partition $\mathcal{P} = \{A_i\}$ of A, define the $\mathcal{C}$-simple maps

$$g = \sum_i a_i C_{A_i} \text{ and } h = \sum_i b_i C_{A_i}$$

with

$$a_i = \inf f[A_i] \text{ and } b_i = \sup f[A_i].$$

Then $g \le f \le h$ on A with

$$\underline{S}(f, \mathcal{P}) = \sum_i a_i\, mA_i = L\int_A g$$

and

$$\overline{S}(f, \mathcal{P}) = \sum_i b_i\, mA_i = L\int_A h.$$

By Theorem 1(c) in §5,

$$\underline{S}(f, \mathcal{P}) = L\int_A g \le L\underline{\int}_A f \le L\overline{\int}_A f \le L\int_A h = \overline{S}(f, \mathcal{P}).$$

As this holds for *any* $\mathcal{P}$, we get

$$R\underline{\int}_A f = \sup_{\mathcal{P}} \underline{S}(f, \mathcal{P}) \le L\underline{\int}_A f \le L\overline{\int}_A f = \inf_{\mathcal{P}} \overline{S}(f, \mathcal{P}) = R\overline{\int}_A f. \tag{5}$$

But by assumption,

$$R\underline{\int}_A f = R\overline{\int}_A f.$$

Thus these inequalities become equations:

$$R\int_A f = \underline{\int}_A f = \overline{\int}_A f = R\int_A f.$$

Also, by definition (c), f is bounded on A; so $|f| < K < \infty$ on A. Hence

$$\left|\int_A f\right| \le \int_A |f| \le K \cdot mA < \infty.^3$$

Thus

$$\underline{\int}_A f = \overline{\int}_A f \neq \pm\infty,$$

i.e., f is Lebesgue integrable, and

$$L\int_A f = R\int_A f,$$

as claimed. □

Note 2. The converse *fails*. For example, as shown in the example in §4, $f = C_R$ ($R =$ rationals) is L-integrable on $A = [0,1]$.

Yet f is *not R-integrable*.

For $\mathcal{C}$-partitions involve intervals containing both rationals (on which $f = 1$) and irrationals (on which $f = 0$). Thus for *any* $\mathcal{P}$,

$$\underline{S}(f,\mathcal{P}) = 0 \text{ and } \overline{S}(f,\mathcal{P}) = 1 \cdot mA = 1.$$

(Why?) So

$$R\overline{\int}_A f = \inf \overline{S}(f,\mathcal{P}) = 1,$$

while

$$R\underline{\int}_A f = 0 \neq R\overline{\int}_A f.$$

Note 3. By Theorem 1, any $R\int_A f$ is also a Lebesgue integral. Thus the rules of §§5–6 apply to R-integrals, *provided that the functions involved are R-integrable*. For a deeper study, we need a few more ideas.

Definitions (continued).

(d) The *mesh* $|\mathcal{P}|$ of a $\mathcal{C}$-partition $\mathcal{P} = \{A_1, \ldots, A_q\}$ is the largest of the diagonals dA_i:

$$|\mathcal{P}| = \max\{dA_1, dA_2, \ldots, dA_q\}.$$

Note 4. For any $A \in \mathcal{C}$, there is a sequence of $\mathcal{C}$-partitions $\mathcal{P}_k$ such that

(i) each P_{k+1} refines P_k and

(ii) $\lim_{k\to\infty} |P_k| = 0$.

---

[3] This also shows that an R-integral, when one exists, is always *finite*.

To construct such a sequence, bisect the edges of A so as to obtain 2^n subintervals of diagonal $\frac{1}{2}dA$ (Chapter 3, §7). Repeat this with each of the subintervals, and so on. Then

$$|P_k| = \frac{dA}{2^k} \to 0.$$

Lemma 3. *Let* $f\colon A \to E^1$ *be bounded. Let* $\{\mathcal{P}_k\}$ *satisfy* (i) *of Note* 4. *If* $P_k = \{A_1^k, \ldots, A_{q_k}^k\}$, *put*

$$g_k = \sum_{i=1}^{q_k} C_{A_i^k} \inf f[A_i^k]$$

and

$$h_k = \sum_{i=1}^{q_k} C_{A_i^k} \sup f[A_i^k].$$

Then the functions

$$g = \sup_k g_k \text{ and } h = \inf_k h_k$$

are Lebesgue integrable on A,[4] *and*

$$\int_A g = \lim_{k\to\infty} \underline{S}(f, \mathcal{P}_k) \le R\underline{\int_A} f \le R\overline{\int_A} f \le \lim_{k\to\infty} \overline{S}(f, \mathcal{P}_k) = \int_A h. \tag{6}$$

Proof. As in Theorem 2, we obtain $g_k \le f \le h_k$ on A with

$$\int_A g_k = \underline{S}(f, \mathcal{P}_k)$$

and

$$\int_A h_k = \overline{S}(f, \mathcal{P}_k).$$

Since $\mathcal{P}_{k+1}$ *refines* $\mathcal{P}_k$, it also easily follows that

$$g_k \le g_{k+1} \le \sup_k g_k = g \le f \le h = \inf_k h_k \le h_{k+1} \le h_k. \tag{7}$$

(Verify!)

Thus $\{g_k\}\uparrow$ and $\{h_k\}\downarrow$, and so

$$g = \sup_k g_k = \lim_{k\to\infty} g_k \text{ and } h = \inf_k h_k = \lim_{k\to\infty} h_k.$$

Also, as f is bounded,

$$(\exists\, K \in E^1) \quad |f| < K \text{ on } A.$$

---

[4] Integrability is with respect to the measure m mentioned above.

The definition of g_k and h_k then implies

$$(\forall\, k) \quad |g_k| \le K \text{ and } |h_k| \le K \text{ (why?)},$$

with

$$\int_A (K) = K \cdot mA < \infty.$$

The g_k and h_k are measurable (even *simple*) on A, with $g_k \to g$ and $h_k \to h$.

Thus by Theorem 5 and Note 1, both from §6, g and h are Lebesgue integrable,[5] with

$$\int_A g = \lim_{k\to\infty} \int_A g_k \text{ and } \int_A h = \lim_{k\to\infty} \int_A h_k.$$

As

$$\int_A g_k = \underline{S}(f, \mathcal{P}_k) \le R \underline{\int}_A f$$

and

$$\int_A h_k = \overline{S}(f, \mathcal{P}_k) \ge R \overline{\int}_A f,$$

passage to the limit in equalities yields (6). Thus the lemma is proved. $\square$

Lemma 4. *With all as in Lemma* 3, *let* B *be the union of the* boundaries *of all intervals from all* $\mathcal{P}_k$. *Let* $|\mathcal{P}_k| \to 0$. *Then we have the following.*

(i) *If* f *is continuous at* $p \in A$, *then* $h(p) = g(p)$.

(ii) *The converse holds if* $p \in A - B$.

Proof. For each k, p is in *one* of the intervals in $\mathcal{P}_k$; call it A_{kp}.

If $p \in A - B$, p is an *interior* point of A_{kp}; so there is a globe

$$G_p(\delta_k) \subseteq A_{kp}.$$

Also, by the definition of g_k and h_k,

$$g_k(p) = \inf f[A_{kp}] \text{ and } h_k = \sup f[A_{kp}].$$

(Why?)

Now fix $\varepsilon > 0$. If $g(p) = h(p)$, then

$$0 = h(p) - g(p) = \lim_{k\to\infty} [h_k(p) - g_k(p)];$$

so

$$(\exists\, k) \quad |h_k(p) - g_k(p)| = \sup f[A_{kp}] - \inf f[A_{kp}] < \varepsilon.$$

As $G_p(\delta_k) \subseteq A_{kp}$, we get

$$(\forall\, x \in G_p(\delta_k)) \quad |f(x) - f(p)| \le \sup f[A_{kp}] - \inf f[A_{kp}] < \varepsilon,$$

---

[5] Integrability is with respect to the measure m mentioned above.

proving continuity (clause (ii)).

For (i), given $\varepsilon > 0$, choose $\delta > 0$ so that

$$(\forall\, x, y \in A \cap G_p(\delta)) \quad |f(x) - f(y)| < \varepsilon.$$

Because

$$(\forall\, \delta > 0)\ (\exists\, k_0)\ (\forall\, k > k_0) \quad |\mathcal{P}_k| < \delta$$

for $k > k_0$, $A_{kp} \subseteq G_p(\delta)$. Deduce that

$$(\forall\, k > k_0) \quad |h_k(p) - g_k(p)| \le \varepsilon. \quad \square$$

Note 5. The *Lebesgue* measure of B in Lemma 4 is zero; for B consists of countably many "*faces*" (degenerate intervals), each of measure zero.

Theorem 3. *A map* $f\colon A \to E^1$ *is R-integrable on* A (*with* $m =$ Lebesgue *measure*) *iff* f *is bounded on* A *and continuous on* $A - Q$ *for some* Q *with* $mQ = 0$.

Note that *relative* continuity on $A - Q$ is *not* enough—take $f = C_R$ of Note 2.

Proof. If these conditions hold, choose $\{\mathcal{P}_k\}$ as in Lemma 4.

Then by the assumed continuity, $g = h$ on $A - Q$, $mQ = 0$.

Thus

$$\int_A g = \int_A h$$

(Corollary 2 in §5).

Hence by formula (6), f is R-integrable on A.

Conversely, if so, use Lemma 1 with

$$\varepsilon = 1, \frac{1}{2}, \ldots, \frac{1}{k}, \ldots$$

to get for each k some $\mathcal{P}_k$ such that

$$\overline{S}(f, \mathcal{P}_k) - \underline{S}(f, \mathcal{P}_k) < \frac{1}{k} \to 0.$$

By Corollary 1, this will still hold if we *refine* each $\mathcal{P}_k$, step by step, so as to achieve properties (i) and (ii) of Note 4 as well. Then Lemmas 3 and 4 apply.

As

$$\overline{S}(f, \mathcal{P}_k) - \underline{S}(f, \mathcal{P}_k) \to 0,$$

formula (6) shows that

$$\int_A g = \lim_{k\to\infty} \underline{S}(f, \mathcal{P}_k) = \lim_{k\to\infty} \overline{S}(f, \mathcal{P}_k) = \int_A h.$$

As h and g are integrable on A,

$$\int_A (h-g) = \int_A h - \int_A g = 0.$$

Also $h - g \geq 0$; so by Theorem 1(h) in §5, $h = g$ on $A - Q'$, $mQ' = 0$ (under *Lebesgue measure*). Hence by Lemma 4, f is continuous on

$$A - Q' - B,$$

with $mB = 0$ (Note 5).

Let $Q = Q' \cup B$. Then $mQ = 0$ and

$$A - Q = A - Q' - B;$$

so f is continuous on $A - Q$. This completes the proof. □

Note 6. The first part of the proof does not involve B and thus works even if m is *not* the Lebesgue measure. *The second part requires that* $mB = 0$.

Theorem 3 shows that R-integrals are limited to *a.e. continuous functions* and hence are less flexible than L-integrals: Fewer functions are R-integrable, and convergence theorems (§6, Theorems 4 and 5) fail unless $R\int_A f$ exists.

III. Functions $f\colon E^n \to E^s$ (C^s). For such functions, R-integrals are defined *componentwise* (see §7). Thus $f = (f_1, \ldots, f_s)$ is R-integrable on A iff all f_k $(k \leq s)$ are, and then

$$R\int_A f = \sum_{k=1}^{s} \bar{e}_k \, R\int_A f_k.$$

A complex function f is R-integrable iff f_{re} and f_{im} are, and then

$$R\int_A f = R\int_A f_{\text{re}} + i\, R\int_A f_{\text{im}}.$$

Via components, Theorems 1 to 3, Corollaries 3 and 4, additivity, linearity, etc., apply.

IV. Stieltjes Integrals. Riemann used *Lebesgue* premeasure v only. But as we saw, his method admits other premeasures, too.

Thus *in* E^1, we may let m be the *LS premeasure* s_α or the *LS measure* m_α, where $\alpha\uparrow$ (Chapter 7, §5, Example (b), and Chapter 7, §9).

Then

$$R\int_A f\, dm$$

is called the *Riemann–Stieltjes* (RS) integral of f *with respect to* α, also written

$$R\int_A f\, d\alpha \quad \text{or} \quad R\int_a^b f(x)\, d\alpha(x)$$

(the latter if $A = [a, b]$); f and α are called the *integrand* and *integrator*, respectively.

If $\alpha(x) = x$, m_α becomes the Lebesgue measure, and

$$R \int f(x)\, d\alpha(x)$$

turns into

$$R \int f(x)\, dx.$$

Our theory still remains valid; only Theorem 3 now reads as follows.

Corollary 4. *If f is bounded and a.e. continuous on $A = [a, b]$ (under an LS measure m_α) then*

$$R \int_a^b f\, d\alpha$$

exists. The converse holds if α is continuous on A.

For by Notes 5 and 6, the "only if" in Theorem 3 holds if $m_\alpha B = 0$. Here B consists of *countably* many *endpoints* of partition subintervals. But (see Chapter 7, §9) $m_\alpha\{p\} = 0$ if α is continuous at p. Thus the later implies $m_\alpha B = 0$.

RS-integration has been used in many fields (e.g., probability theory, physics, etc.), but it is superseded by LS-integration, i.e., Lebesgue integration with respect to m_α, which is fully covered by the general theory of §§1–8.

Actually, Stieltjes himself used somewhat different definitions (see Problems 10–13), which amount to applying the *set function* σ_α of Problem 9 in Chapter 7, §4, instead of s_α or m_α. We reserve the name "*Stieltjes* integrals," denoted

$$S \int_a^b f\, d\alpha,$$

for *such* integrals, and "*RS-integrals*" for those based on m_α or s_α (this terminology is not standard).

Observe that σ_α *need not be* ≥ 0. Thus for the first time, we encounter integration with respect to *sign-changing* set functions. A much more general theory is presented in §10 (see Problem 10 there).

Problems on Riemann and Stieltjes Integrals

1. Replacing "$\mathcal{M}$" by "$\mathcal{C}$," and "elementary and integrable" or "elementary and nonnegative" by "$\mathcal{C}$-simple," prove Corollary 1(ii)(iv)(vii) and Theorems 1(i) and 2(ii), all in §4, and do Problem 5–7 in §4, for R-integrals.

2. Verify Note 1.

2′. Do Problems 5–7 in §5 for R-integrals.

3. Do the following for R-integrals.

(i) Prove Theorems 1(a)–(g) and 2, both in §5 ($\mathcal{C}$-partitions only).

(ii) Prove Theorem 1 and Corollaries 1 and 2, all in §6.

(iii) Show that definition (b) can be replaced by formulas analogous to formulas (1′), (1″), and (1) of Definition 1 in §5.

[Hint: Use Problems 1 and 2′.]

4. Fill in all details in the proof of Theorem 1, Lemmas 3 and 4, and Corollary 4.

5. For $f, g \colon E^n \to E^s\,(C^s)$, via components, prove the following.

(i) Theorems 1–3 and

(ii) additivity and linearity of R-integrals.

Do also Problem 13 in §7 for R-integrals.

6. Prove that if $f \colon A \to E^s\,(C^s)$ is bounded and a.e. continuous on A, then

$$R\int_A |f| \geq \left| R\int_A f \right|.$$

For $m =$ Lebesgue measure, do it assuming R-integrability only.

7. Prove that if $f, g \colon A \to E^1$ are R-integrable, then

(i) so is f^2, and

(ii) so is fg.

[Hints: (i) Use Lemma 1. Let $h = |f| \leq K < \infty$ on A. Verify that

$$(\inf h[A_i])^2 = \inf f^2[A_i] \text{ and } (\sup h[A_i])^2 = \sup f^2[A_i];$$

so

$$\begin{aligned}\sup f^2[A_i] - \inf f^2[A_i] &= (\sup h[A_i] + \inf h[A_i])\,(\sup h[A_i] - \inf h[A_i])\\ &\leq (\sup h[A_i] - \inf h[A_i])\,2K.\end{aligned}$$

(ii) Use

$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2].$$

(iii) For $m =$ Lebesgue measure, do it using Theorem 3.]

8. Prove that if $m =$ the volume function v (or LS function s_α for a *continuous* α), then in formulas (1) and (2), one may replace A_i by $\overline{A}_i$ (closure of A_i).

[Hint: Show that here $mA = m\overline{A}$,

$$R\int_A f = R\int_{\overline{A}} f,$$

and additivity works even if the A_i have some common "faces" (only their *interiors* being disjoint).]

9. (Riemann sums.) Instead of $\underline{S}$ and $\overline{S}$, Riemann used sums

$$S(f,\mathcal{P}) = \sum_i f(x_i)\, dmA_i,$$

where $m = v$ (see Problem 8) and x_i is arbitrarily chosen from $\overline{A_i}$.

For a bounded f, prove that

$$r = R\int_A f\, dm$$

exists on $A = [a,b]$ iff for every $\varepsilon > 0$, there is $\mathcal{P}_\varepsilon$ such that

$$|S(f,\mathcal{P}) - r| < \varepsilon$$

for every *refinement*

$$\mathcal{P} = \{A_i\}$$

of $\mathcal{P}_\varepsilon$ and *any* choice of $x_i \in \overline{A_i}$.

[Hint: Show that by Problem 8, this is equivalent to formula (3).]

10. Replacing m by the σ_α *of Problem* 9 *of Chapter* 7, §4, write $S(f,\mathcal{P},\alpha)$ for $S(f,\mathcal{P})$ in Problem 9, treating Problem 9 as a *definition* of the *Stieltjes integral*,

$$S\int_a^b f\, d\alpha \quad \left(\text{or } S\int_a^b f\, d\sigma_\alpha\right).$$

Here $f, \alpha\colon E^1 \to E^1$ (monotone or not; even $f, \alpha\colon E^1 \to C$ will do).

Prove that if $\alpha\colon E^1 \to E^1$ is continuous and $\alpha\!\uparrow$, then

$$S\int_a^b f\, d\alpha = R\int_a^b f\, d\alpha,$$

the RS-integral.

11. (Integration by parts.) Continuing Problem 10, prove that

$$S\int_a^b f\, d\alpha$$

exists iff

$$S\int_a^b \alpha\, df$$

does, and then

$$S\int_a^b f\, d\alpha + S\int_a^b \alpha\, df = K,$$

where

$$K = f(b)\,\alpha(b) - f(a)\,\alpha(a).$$

[Hints: Take any $\mathcal{C}$-partition $\mathcal{P} = \{A_i\}$ of $[a, b]$, with

$$\overline{A_i} = [y_{i-1}, y_i],$$

say. For any $x_i \in \overline{A}_i$, verify that

$$S(f, \mathcal{P}, \alpha) = \sum f(x_i)\,[\alpha(y_i) - \alpha(y_{i-1})] = \sum f(x_i)\,\alpha(y_i) - \sum f(x_i)\,\alpha(y_{i-1})$$

and

$$K = \sum f(x_i)\,\alpha(y_i) - \sum f(x_{i-1})\,\alpha(y_{i-1}).$$

Deduce that

$$K - S(f, \mathcal{P}, \alpha) = S(\alpha, \mathcal{P}', f) = \sum \alpha(x_i)\,[f(x_i) - f(y_i)] - \sum \alpha(x_{i-1})\,[f(y_i) - f(x_{i-1})];$$

here $\mathcal{P}'$ results by *combining* the partition points x_i and y_i, so it *refines* $\mathcal{P}$.

Now, if $S \int_a^b \alpha\, df$ exists, fix $\mathcal{P}_\varepsilon$ as in Problem 9 and show that

$$\left| K - S(f, \mathcal{P}, \alpha) - S \int_a^b \alpha\, df \right| < \varepsilon$$

whenever $\mathcal{P}$ *refines* $\mathcal{P}_\varepsilon$.]

12. If $\alpha \colon E^1 \to E^1$ is of class CD^1 on $[a, b]$ and if

$$S \int_a^b f\, d\alpha$$

exists (see Problem 10), it equals

$$R \int_a^b f(x)\,\alpha'(x)\,dx.$$

[Hints: Set $\phi = f\,\alpha'$, $\mathcal{P} = \{A_i\}$, $\overline{A_i} = [a_{i-1}, a_i]$. Then

$$S(\phi, \mathcal{P}) = \sum f(x_i)\,\alpha'(x_i)\,(a_i - a_{i-1}), \quad x_i \in \overline{A_i},$$

and (Corollary 3 in Chapter 5, §2)

$$S(f, \mathcal{P}, \alpha) = \sum f(x_i)\,[\alpha(a_i) - \alpha(a_{i-1})] = \sum f(x_i)\,\alpha'(q_i), \quad q_i \in A_i.$$

As f is bounded and α' is *uniformly* continuous on $[a, b]$ (why?), deduce that

$(\forall\, \varepsilon > 0)\ (\exists\, \mathcal{P}_\varepsilon)\ (\forall\, \mathcal{P}$ refining $\mathcal{P}_\varepsilon)$

$$|S(\phi, \mathcal{P}) - S(f, \mathcal{P}, \alpha)| < \frac{1}{2}\varepsilon \text{ and } \left| S(f, \mathcal{P}, \alpha) - S \int_a^b f\, d\alpha \right| < \frac{1}{2}\varepsilon.$$

Proceed. Use Problem 9.]

13. (Laws of the mean.) Let $f, g, \alpha \colon E^1 \to E^1$; $p \le f \le q$ on $A = [a, b]$; $p, q \in E^1$. Prove the following.

(i) If $\alpha\!\uparrow$ and if

$$S \int_a^b f\, d\alpha$$

exists, then $(\exists\, c \in [p, q])$ such that

$$S \int_a^b f\, d\alpha = c\,[\alpha(b) - \alpha(a)].$$

Similarly, if

$$R \int_a^b f\, d\alpha$$

exists, then $(\exists\, c \in [p, q])$ such that

$$R \int_a^b f\, d\alpha = c\,[\alpha(b+) - \alpha(a-)].$$

(i′) If f also has the Darboux property on A, then $c = f(x_0)$ for some $x_0 \in A$.

(ii) If α is continuous, and $f\!\uparrow$ on A, then

$$S \int_a^b f\, d\alpha = [f(b)\,\alpha(b) - f(a)\,\alpha(a)] - S \int_a^b \alpha\, df$$

exists, and $(\exists\, z \in A)$ such that

$$\begin{aligned} S \int_a^b f\, d\alpha &= f(a)\, S \int_a^z d\alpha + f(b)\, S \int_z^b d\alpha \\ &= f(a)\,[\alpha(z) - \alpha(a)] + f(b)\,[\alpha(b) - \alpha(z)]. \end{aligned}$$

(ii′) If g is continuous and $f\!\uparrow$ on A, then $(\exists\, z \in A)$ such that

$$R \int_a^b f(x)\, g(x)\, dx = p \cdot R \int_a^z g(x)\, dx + q \cdot R \int_z^b g(x)\, dx.$$

If $f\!\downarrow$, replace f by $-f$. (See also Corollary 5 in Chapter 9, §1.)

[Hints: (i) As $\alpha\!\uparrow$, we get

$$p\,[\alpha(b) - \alpha(a)] \le S \int_a^b f\, d\alpha \le q\,[\alpha(b) - \alpha(a)].$$

(Why?) Now argue as in §6, Theorem 3 and Problem 2.

(ii) Use Problem 11, and apply (i) to $\int \alpha\, df$.

(ii′) By Theorem 2 of Chapter 5, §10, g has a *primitive* $\beta \in CD^1$. Apply Problem 12 to $S \int_a^b f\, d\beta$.]

§10. Integration in Generalized Measure Spaces

Let $(S, \mathcal{M}, s)$ be a generalized measure space. By Note 1 in §3, a map f is s-measurable iff it is v_s-measurable. This naturally leads us to the following definition.

Definition 1.

A map $f\colon S \to E$ is s-integrable on a set A iff it is v_s-integrable on A. (Recall that v_s, the total variation of s, is a *measure*.)

Note 1. Here the range spaces of f and s are assumed *complete* and such that $f(x)\,sA$ is *defined* for $x \in S$ and $A \in \mathcal{M}$. Thus if s is vector valued, f must be scalar valued, and vice versa. Later, if a factor p occurs, it must be such that $p\,f(x)\,sA$ is defined, i.e., at least *two* of p, $f(x)$, and sA are scalars.

Note 2. If s is a *measure* (≥ 0), then $v_s = s^+ = s$ (Corollary 3 in Chapter 7, §11); so our present definition agrees with the previous ones (as in Theorem 1 of §7).

Lemma 1. *If m' and m'' are measures, with $m' \geq m''$ on $\mathcal{M}$, then*

$$\int_A |f|\,dm' \geq \int_A |f|\,dm''$$

for all $A \in \mathcal{M}$ and any $f\colon S \to E$.

Proof. First, take any *elementary and nonnegative* map $g \geq |f|$,

$$g = \sum_i C_{A_i} a_i \text{ on } A.$$

Then (§4)

$$\int_A g\,dm' = \sum a_i\,m'A_i \geq \sum a_i\,m''A_i = \int_A g\,dm''.$$

Hence by Definition 1 in §5,

$$\int_A |f|\,dm' = \inf_{g \geq |f|} \int_A g\,dm' \geq \inf_{g \geq |f|} \int_A g\,dm'' = \int_A |f|\,dm'',$$

as claimed. □

Lemma 2.

(i) *If $s\colon \mathcal{M} \to E^n\,(C^n)$ with $s = (s_1, \ldots, s_n)$, and if f is s-integrable on $A \in \mathcal{M}$, then f is s_k-integrable on A for $k = 1, 2, \ldots, n$.*

(ii) *If s is a* signed *measure and f is s-integrable on A, then f is integrable on A with respect to both s^+ and s^- (with s^+ and s^- as in formula (3) in Chapter* 7, §11).

Note 3. The converse statements hold if f is $\mathcal{M}$-measurable on A.

Proof.

(i) If $s = (s_1, \ldots, s_n)$, then (Problem 4 of Chapter 7, §11)

$$v_s \geq v_{s_k}, \quad k = 1, \ldots, n.$$

Hence by Definition 1 and Lemma 1, the s-integrability of f implies

$$\infty > \int_A |f| \, dv_s \geq \int_A |f| \, dv_{s_k}.$$

Also, f is v_s-measurable, i.e., $\mathcal{M}$-measurable on $A - Q$, with

$$0 = v_s Q \geq v_{s_k} Q \geq 0.$$

Thus f is s_k-integrable on A, $k = 1, \ldots, n$, as claimed.

(ii) If $s = s^+ - s^-$, then by Theorem 4 in Chapter 7, §11, and Corollary 3 there, s^+ and s^- are *measures* (≥ 0) and $v_s = s^+ + s^-$, so that both

$$v_s \geq s^+ = v_{s^+} \text{ and } v_s \geq s^- = v_{s^-}.$$

Thus the desired result follows exactly as in part (i) of the proof. □

We leave Note 3 as an exercise.

Definition 2.

If f is s-integrable on $A \in \mathcal{M}$, we set

(i) in the case $s\colon \mathcal{M} \to E^*$,

$$\int_A f \, ds = \int_A f \, ds^+ - \int_A f \, ds^-,$$

with s^+ and s^- as in formula (3) of Chapter 7, §11;[1]

(ii) in the case $s\colon \mathcal{M} \to E^n \, (C^n)$,

$$\int_A f \, ds = \sum_{k=1}^{n} \vec{e}_k \int_A f \, ds_k,$$

with $\vec{e}_k$ as in Theorem 2 of Chapter 3, §§1–3;

(iii) if $s\colon \mathcal{M} \to C$,

$$\int_A f \, ds = \int_A f \, ds_{\mathrm{re}} + i \cdot \int_A f \, ds_{\mathrm{im}}.$$

(See also Problems 2 and 3.)

Note 4. If s is a *measure*, then

$$s = s^+ = s_{\mathrm{re}} = s_1$$

and

$$0 = s^- = s_{\mathrm{im}} = s_2;$$

---

[1] By choosing s^+ and s^- as in formula (3) of Chapter 7, §11, we avoid ambiguity.

so Definition 2 agrees with our previous definitions. Similarly for $s\colon \mathcal{M} \to E^n\,(C^n)$.

Below, s, t, and u are generalized measures on $\mathcal{M}$ *as in Definition* 2, while $f, g\colon S \to E$ are functions, with E a complete normed space, as in Note 1.

Theorem 1. *The linearity, additivity, and σ-additivity properties* (*as in* §7, *Theorems* 2 *and* 3) *also apply to integrals*

$$\int_A f\,ds,$$

with s as in Definition 2.

Proof. (i) *Linearity*: Let $f, g\colon S \to E$ be s-integrable on $A \in \mathcal{M}$. Let p, q be suitable constants (see Note 1).

If s is a *signed* measure, then by Lemma 2(ii) and Definitions 1 and 2, f is integrable with respect to v_s, s^+, and s^-. As these are *measures*, Theorem 2 in §7 shows that $pf + qg$ is integrable with respect to v_s, s^+, and s^-, and by Definition 2,

$$\begin{aligned}\int_A (pf+qg)\,ds &= \int_A (pf+qg)\,ds^+ - \int_A (pf+qg)\,ds^- \\ &= p\int_A f\,ds^+ + q\int_A g\,ds^+ - p\int_A f\,ds^- - q\int_A g\,ds^- \\ &= p\int_A f\,ds + q\int_A g\,ds.\end{aligned}$$

Thus linearity holds for *signed* measures. Via components, it now follows for $s\colon \mathcal{M} \to E^n\,(C^n)$ as well. Verify!

(ii) *Additivity and σ-additivity* follow in a similar manner. □

Corollary 1. *Assume f is s-integrable on A, with s as in Definition* 2.

(i) *If f is constant* ($f = c$) *on A, we have*

$$\int_A f\,ds = c \cdot sA.$$

(ii) *If*

$$f = \sum_i a_i\,C_{A_i}$$

for an $\mathcal{M}$-partition $\{A_i\}$ of A, then

$$\int_A f\,ds = \sum_i a_i\,sA_i \text{ and } \int_A |f|\,ds = \sum_i |a_i|\,sA_i$$

(*both series absolutely convergent*).

(iii) *$|f| < \infty$ a.e. on A.*[2]

(iv) *$\int_A |f| \, dv_s = 0$ iff $f = 0$ a.e. on A.*

(v) *The set $A(f \neq 0)$ is (v_s) σ-finite (Definition 4 in Chapter 7, §5).*

(vi) *$\int_A f \, ds = \int_{A-Q} f \, ds$ if $v_s Q = 0$ or $f = 0$ on Q ($Q \in \mathcal{M}$).*

(vii) *f is s-integrable on any $\mathcal{M}$-set $B \subseteq A$.*

Proof.

(i) If $s = s^+ - s^-$ is a *signed* measure, we have by Definition 2 that

$$\int_A f \, ds = \int_A f \, ds^+ - \int_A f \, ds^- = c\,(s^+A - s^-A) = c \cdot sA,$$

as required.

For $s\colon \mathcal{M} \to E^n$ (C^n), the result now follows via components. (Verify!)

(ii) As $f = a_i$ on A_i, clause (i) yields

$$\int_{A_i} f \, ds = a_i \, sA_i, \quad i = 1, 2, \ldots .$$

Hence by σ-additivity,

$$\int_A f \, ds = \sum_i \int_{A_i} f \, ds = \sum_i a_i \, sA_i,$$

as claimed.

Clauses (iii), (iv), and (v) follow by Corollary 1 in §5 and Theorem 1(b)(h) there, as v_s is a *measure*; (vi) is proved as §5, Corollary 2. We leave (vii) as an exercise. □

Theorem 2 (dominated convergence). *If*

$$f = \lim_{i \to \infty} f_i \ (\textit{pointwise})$$

on $A - Q$ ($v_s Q = 0$) and if each f_i is s-integrable on A, so is f, and

$$\int_A f \, ds = \lim_{i \to \infty} \int_A f_i \, ds,$$

all provided that

$$(\forall\, i) \quad |f_i| \leq g$$

for some map g with $\int_A g \, dv_s < \infty$.

Proof. If s is a *measure*, this follows by Theorem 5 in §6. Thus as v_s *is* a measure, f is v_s-integrable (hence s-integrable) on A, as asserted.

---

[2] That is, on $A - Q$, $v_s Q = 0$.

Next, if $s = s^+ - s^-$ is a *signed* measure, Lemma 2 shows that f and the f_i are s^+ and s^--integrable as well, with

$$\int_A |f_i|\,ds^+ \leq \int_A |f_i|\,dv_s \leq \int_A g\,dv_s < \infty;$$

similarly for

$$\int_A |f_i|\,ds^-.$$

As s^+ and s^- are *measures*, Theorem 5 of §6 yields

$$\int_A f\,ds = \int_A f\,ds^+ - \int_A f\,ds^- = \lim\left(\int_A f_i\,ds^+ - \int_A f_i\,ds^-\right) = \lim \int_A f_i\,ds.$$

Thus all is proved for *signed* measures.

In the case $s\colon \mathcal{M} \to E^n\,(C^n)$, the result now easily follows by Definition 2(ii)(iii) via components. □

Theorem 3 (uniform convergence)**.** *If $f_i \to f$ (uniformly) on $A - Q$ ($v_s A < \infty$, $v_s Q = 0$), and if each f_i is s-integrable on A, so is f, and*

$$\int_A f\,ds = \lim_{i\to\infty} \int_A f_i\,ds.$$

Proof. Argue as in Theorem 2, replacing §6, Theorem 5, by §7, Lemma 1. □

Our next theorem shows that integrals behave *linearly with respect to measures.*

Theorem 4. *Let $t, u\colon \mathcal{M} \to E^*\,(E^n, C^n)$, with $v_t < \infty$ on $\mathcal{M}$,*[3] *and let*

$$s = pt + qu$$

for finite constants p and q. Then the following statements are true.

(a) *If t and u are generalized measures, so is s.*

(b) *If, further, f is $\mathcal{M}$-measurable on a set A and is both t- and u-integrable on A, it is also s-integrable on A, and*

$$\int_A f\,ds = p\int_A f\,dt + q\int_A f\,du.$$

Proof. We consider only assertion (b) for $s = t + u$; the rest is easy.

First, let f be $\mathcal{M}$-*elementary* on A. By Corollary 1(ii), we set

$$\int_A f\,dt = \sum_i a_i\,tA_i \text{ and } \int_A f\,du = \sum_i a_i\,uA_i.$$

---

[3] Or $|t| < \infty$; see Theorem 6 in Chapter 7, §11. The restriction is redundant if $t\colon \mathcal{M} \to E^n\,(C^n)$.

Also, by integrability,

$$\infty > \int_A |f|\,dv_t = \sum |a_i|\,v_t A_i \text{ and } \infty > \int_A |f|\,dv_u = \sum_i |a_i|\,v_u A_i.$$

Now, by Problem 4 in Chapter 7, §11,

$$v_s = v_{t+u} \leq v_t + v_u;$$

so

$$\begin{aligned}\int_A |f|\,dv_s &= \sum_i |a_i|\,v_s A_i \\ &\leq \sum_i |a_i|\,(v_t A_i + v_u A_i) = \int_A |f|\,dv_t + \int_A |f|\,dv_u < \infty.\end{aligned}$$

As f is also $\mathcal{M}$-measurable (even elementary), it is s-integrable on A (by Definition 1), and

$$\int_A f\,ds = \sum_i a_i\,sA_i = \sum_i a_i\,(tA_i + uA_i) = \int_A f\,dt + \int_A f\,du,$$

as claimed.

Next, suppose f is $\mathcal{M}$-measurable on A and $v_u A < \infty$. By assumption, $v_t A < \infty$, too; so

$$v_s A \leq v_t A + v_u A < \infty.$$

Now, by Theorem 3 in §1,

$$f = \lim_{i \to \infty} f_i \text{ (uniformly)}$$

for some $\mathcal{M}$-*elementary* maps f_i on A. By Lemma 2 in §7, for large i, the f_i are integrable with respect to both v_t and v_u on A. By what was shown above, they are also s-integrable, with

$$\int_A f_i ds = \int_A f_i dt + \int_A f_i\,du.$$

With $i \to \infty$, Theorem 3 yields the result.

Finally, let $v_u A = \infty$. By Corollary 1(v), we may assume (as in Lemma 3 of §7) that $A_i \nearrow A$, with $v_u A_i < \infty$, and $v_t A_i < \infty$ (since $v_t < \infty$, by assumption). Set

$$f_i = f\,C_{A_i} \to f \text{ (pointwise)}$$

on A, with $|f_i| \leq |f|$. (Why?)

As $f_i = f$ on A_i and $f_i = 0$ on $A - A_i$, all f_i are both t- and u-integrable *on* A (for f is). Since $v_t A_i < \infty$ and $v_u A_i < \infty$, the f_i are also s-integrable (as shown above), with

$$\int_A f_i\,ds = \int_{A_i} f_i\,ds = \int_{A_i} f_i\,dt + \int_{A_i} f_i\,du = \int_A f_i\,dt + \int_A f_i\,du.$$

With $i \to \infty$, Theorem 2 now yields the result.

To complete the proof of (b), it suffices to consider, along similar lines, the case $s = pt$ (or $s = qu$). We leave this to the reader.

For (a), see Chapter 7, §11. □

Theorem 5. *If f is s-integrable on A, so is $|f|$, and*

$$\left|\int_A f\,ds\right| \le \int_A |f|\,dv_s.$$

Proof. By Definition 1, and Theorem 1 of §1, f and $|f|$ are $\mathcal{M}$-measurable on $A - Q$, $v_s Q = 0$, and

$$\int_A |f|\,dv_s < \infty;$$

so $|f|$ is s-integrable on A.

The desired inequality is immediate by Corollary 1(ii) if f is elementary.

Next, exactly as in Theorem 4, one obtains it for the case $v_s A < \infty$, and then for $v_s A = \infty$. We omit the details. □

Definition 3.

We write

$$\text{“}ds = g\,dt \textit{ in } A\text{”}$$

or

$$\text{“}s = \int g\,dt \textit{ in } A\text{”}$$

iff g is t-integrable on A, and

$$sX = \int_X g\,dt$$

for $A \supseteq X$, $X \in \mathcal{M}$.

We then call s the *indefinite integral* of g *in* A. ($\int_X g\,dt$ may be interpreted as in Problems 2–4 below.)

Lemma 3. *If $A \in \mathcal{M}$ and*

$$ds = g\,dt \textit{ in } A,$$

then

$$dv_s = |g|\,dv_t \textit{ in } A.$$

Proof. By assumption, g and $|g|$ are v_t-integrable on X, and

$$sX = \int_X g\,dt$$

for $A \supseteq X$, $X \in \mathcal{M}$. We must show that

$$v_s X = \int_X |g|\,dv_t$$

for such X.

This is easy if $g = c$ (constant) on X. For by definition,

$$v_s X = \sup_{\mathcal{P}} \sum_i |sX_i|,$$

over all $\mathcal{M}$-partitions $\mathcal{P} = \{X_i\}$ of X. As

$$sX_i = \int_{X_i} g\,dt = c \cdot tX_i,$$

we have

$$v_s X = \sup_{\mathcal{P}} \sum_i |c|\,|tX_i| = |c| \sup_{\mathcal{P}} \sum_i |tX_i| = |c|\,v_t X;$$

so

$$v_s X = \int_X |g|\,dv_t.$$

Thus all is proved for *constant* g.

Hence by σ-additivity, the lemma holds for $\mathcal{M}$-*elementary maps* g. (Why?)

In the general case, g is t-integrable on X, hence $\mathcal{M}$-measurable and finite on $X - Q$, $v_t Q = 0$. By Corollary 1(iii), we may assume g finite and measurable *on* X; so

$$g = \lim_{k\to\infty} g_k \text{ (uniformly)}$$

on X for some $\mathcal{M}$-elementary maps g_k, all integrable on X, with respect to v_t (and t).

Let

$$s_k = \int g_k\,dt$$

in X. By what we just proved for elementary and integrable maps,

$$v_{s_k} X = \int_X |g_k|\,dv_t, \quad k = 1, 2, \ldots .$$

Now, if $v_t X < \infty$, Theorem 3 yields

$$\int_X |g|\,dv_t = \lim_{k\to\infty} \int_X |g_k|\,dv_t = \lim_{k\to\infty} v_{s_k} X = v_s X$$

(see Problem 6). Thus all is proved if $v_t X < \infty$.

If, however, $v_t X = \infty$, argue as in Theorem 4 (the *last* step), using the left continuity of v_s and of

$$\int |g| \, dv_t.$$

Verify! □

Theorem 6 (change of measure). *If f is s-integrable on $A \in \mathcal{M}$, with*

$$ds = g \, dt \text{ in } A,$$

then (subject to Note 1) fg is t-integrable on A and

$$\int_A f \, ds = \int_A fg \, dt.$$

(Note the formal substitution of "$g\,dt$" for "ds.")

Proof. The proof is easy if f is constant or elementary on A (use Corollary 1(ii)). We leave this case to the reader, and next we assume g is bounded and $v_t A < \infty$.

By s-integrability, f is $\mathcal{M}$-measurable and finite on $A - Q$, with

$$0 = v_s Q = \int_Q |g| \, dv_t$$

by Lemma 3. Hence $0 = g = fg$ on $Q - Z$, $v_t Z = 0$. Therefore,

$$\int_Q fg \, dt = 0 = \int_Q f \, ds$$

for $v_s Q = 0$. Thus we may neglect Q and assume that f is finite and $\mathcal{M}$-measurable *on* A.

As $ds = g \, dt$, Definition 3 and Lemma 3 yield

$$v_s A = \int_A |g| \, dv_t < \infty.$$

Also (Theorem 3 in Chapter 8, §1),

$$f = \lim_{k \to \infty} f_k \quad \text{(uniformly)}$$

for elementary maps f_k, all v_s-integrable on A (Lemma 2 in §7). As g is *bounded*, we get on A

$$fg = \lim_{k \to \infty} f_k g \quad \text{(uniformly)}.$$

Moreover, as the theorem holds for *elementary and integrable* maps, $f_k g$ is t-integrable on A, and

$$\int_A f_k\, ds = \int_A f_k g\, dt, \quad k = 1, 2, \ldots .$$

Since $v_s A < \infty$ and $v_t A < \infty$, Theorem 3 shows that fg is t-integrable on A, and

$$\int_A f\, ds = \lim_{k\to\infty} \int_A f_k\, ds = \lim_{k\to\infty} \int_A f_k g\, dt = \int_A fg\, dt.$$

Thus all is proved if $v_t A < \infty$ and g is bounded on A.

In the general case, we again drop a null set to make f and g finite and $\mathcal{M}$-measurable on A. By Corollary 1(v), we may again assume $A_i \nearrow A$, with $v_t A_i < \infty$ $(\forall\, i)$.

Now for $i = 1, 2, \ldots$ set

$$g_i = \begin{cases} g & \text{on } A_i(|g| \le i), \\ 0 & \text{elsewhere.} \end{cases}$$

Then each g_i is *bounded*,

$$g_i \to g \text{ (pointwise)},$$

and

$$|g_i| \le |g|$$

on A. We also set $f_i = f\, C_{A_i}$; so $f_i \to f$ (pointwise) and $|f_i| \le |f|$ on A. Then

$$\int_A f_i\, ds = \int_{A_i} f_i\, ds = \int_{A_i} f_i g_i\, dt = \int_A f_i g_i\, dt.$$

(Why?) Since $|f_i g_i| \le |fg|$ and $f_i g_i \to fg$, the result follows by Theorem 2. □

Problems on Generalized Integration

Recall that in this section E is assumed to be a *complete* normed space.

1. Fill in the missing details in the proofs of this section. Prove Note 3.

2. Treat Corollary 1(ii) as a *definition* of

$$\int_A f ds$$

for $s\colon \mathcal{M} \to E$ and *elementary and integrable* f, even if $E \neq E^n$ (C^n). Hence deduce Corollary 1(i)(vi) for this more general case.

3. Using Lemma 2 in §7, with $m = v_s$, $s\colon \mathcal{M} \to E$, construct

$$\int_A f\, ds$$

as in Definition 2 of §7 for the case $v_s A \neq \infty$. Show that this agrees with Problem 2 if f is *elementary and integrable*. Then prove linearity *for functions with v_s-finite support* as in §7.

4. Define

$$\int_A f\,ds \quad (s\colon \mathcal{M} \to E)$$

also for $v_s A = \infty$.

[Hint: Set $m = v_s$ in Lemma 3 of §7.]

5. Prove Theorems 1 to 3 for the *general* case, $s\colon \mathcal{M} \to E$ (see Problem 4).

[Hint: Argue as in §7.]

5′. From Problems 2–4, deduce Definition 2 as a *theorem* in the case $E = E^n$ (C^n).

6. Let $s, s_k\colon \mathcal{M} \to E$ $(k = 1, 2, \dots)$ be *any* set functions. Let $A \in \mathcal{M}$ and

$$\mathcal{M}_A = \{X \in \mathcal{M} \mid X \subseteq A\}.$$

Prove that if

$$(\forall\, X \in \mathcal{M}_A) \qquad \lim_{k\to\infty} s_k X = sX,$$

then

$$\lim_{k\to\infty} v_{s_k} A = v_s A,$$

provided $\lim\limits_{k\to\infty} v_{s_k}$ exists.

[Hint: Using Problem 2 in Chapter 7, §11, fix a *finite* disjoint sequence $\{X_i\} \subseteq \mathcal{M}_A$. Then

$$\sum_i |sX_i| = \sum_i \lim_{k\to\infty} |s_k X_i| = \lim_{k\to\infty} \sum_i |s_k X_i| \leq \lim_{k\to\infty} v_{s_k} A.$$

Infer that

$$v_s A \leq \lim_{k\to\infty} v_{s_k} A.$$

Also,

$$(\forall\, \varepsilon > 0)\ (\exists\, k_0)\ (\forall\, k > k_0) \quad \sum_i |s_k X_i| \leq \sum_i |sX_i| + \varepsilon \leq v_s A + \varepsilon.$$

Proceed.]

7. Let $(X, \mathcal{M}, m)$ and $(Y, \mathcal{N}, n)$ be two generalized measure spaces $(X \in \mathcal{M},\ Y \in \mathcal{N})$ such that mn is *defined* (Note 1). Set

$$\mathcal{C} = \{A \times B \mid A \in \mathcal{M},\ B \in \mathcal{N},\ v_m A < \infty,\ v_n B < \infty\}$$

and $s(A \times B) = mA \cdot nB$ for $A \times B \in \mathcal{C}$.

Define a *Fubini* map as in §8, *Part* IV, dropping, however, $\int_{X\times Y} f\,dp$ from Fubini property (c) temporarily.

Then prove Theorem 1 in §8, including formula (1), for Fubini maps so modified.

[Hint: For σ-additivity, use our present Theorem 2 *twice*. Omit $\mathcal{P}^*$.]

8. Continuing Problem 7, let $\mathcal{P}$ be the σ-ring generated by $\mathcal{C}$ in $X \times Y$. Prove that $(\forall D \in \mathcal{P})$ C_D is a Fubini map (as modified).

[Outline: Proceed as in Lemma 5 of §8. For step (ii), use Theorem 2 in §10 *twice*.]

9. Further continuing Problems 7 and 8, *define*

$$(\forall D \in \mathcal{P}) \quad pD = \int_X \int_Y C_D \, dn \, dm.$$

Show that p is a generalized measure, with $p = s$ on $\mathcal{C}$, and that

$$(\forall D \in \mathcal{P}) \quad pD = \int_{X \times Y} C_D \, dp,$$

with the following convention: If $X \times Y \notin \mathcal{P}$, we set

$$\int_{X \times Y} f \, dp = \int_H f \, dp$$

whenever $H \in \mathcal{P}$, f is p-integrable on H, and $f = 0$ on $-H$.

Verify that this is unambiguous, i.e.,

$$\int_{X \times Y} f \, dp$$

so defined is independent of the choice of H.

Finally, let $\overline{p}$ be the *completion* of p (Chapter 7, §6, Problem 15); let $\mathcal{P}^*$ be its domain.

Develop the rest of Fubini theory "imitating" Problem 12 in §8.

10. *Signed Lebesgue–Stieltjes (LS) measures* in E^1 are defined as shown in Chapter 7, §11, Part V. Using the notation of that section, prove the following:

(i) Given a Borel–Stieltjes measure σ_α^* in an interval $I \subseteq E^1$ (or an LS measure $s_\alpha = \overline{\sigma^*}_\alpha$ in I), there are two monotone functions $g\uparrow$ and $h\uparrow$ $(\alpha = g - h)$ such that

$$m_g = s_\alpha^+ \text{ and } m_h = s_\alpha^-,$$

both satisfying formula (3) of Chapter 7, §11, inside I.

(ii) If f is s_α-integrable on $A \subseteq I$, then

$$\int_A f \, ds_\alpha = \int_A f \, dm_g - \int_A f \, dm_h$$

for *any* $g\uparrow$ and $h\uparrow$ (finite) such that $\alpha = g - h$.

[Hints: (i) Define s_α^+ and s_α^- by formula (3) of Chapter 7, §11. Then arguing as in Theorem 2 in Chapter 7, §9, find $g\uparrow$ and $h\uparrow$ with $m_g = s_\alpha^+$ and $m_h = s_\alpha^-$.

(ii) First let $A = (a, b] \subseteq I$, then $A \in \mathcal{B}$. Proceed.]

*§11. The Radon–Nikodym Theorem. Lebesgue Decomposition

I. As you know, the indefinite integral

$$\int f\,dm$$

is a *generalized measure*. We now seek conditions under which a *given* generalized measure μ can be represented as

$$\mu = \int f\,dm$$

for some f (to be found). We start with two lemmas.

Lemma 1. *Let $m, \mu \colon \mathcal{M} \to [0, \infty)$ be* finite *measures in S. Suppose $S \in \mathcal{M}$, $\mu S > 0$ (i.e., $\mu \not\equiv 0$) and μ is m-continuous* (*Chapter* 7, §11).

Then there is $\delta > 0$ and a set $P \in \mathcal{M}$ such that $mP > 0$ and

$$(\forall X \in \mathcal{M}) \quad \mu X \geq \delta \cdot m(X \cap P).$$

Proof. As $m < \infty$ and $\mu S > 0$, there is $\delta > 0$ such that

$$\mu S - \delta \cdot mS > 0.$$

Fix such a δ and define a *signed measure* (Lemma 2 of Chapter 7, §11)

$$\Phi = \mu - \delta m,$$

so that

$$(\forall Y \in \mathcal{M}) \quad \Phi Y = \mu Y - \delta \cdot mY; \tag{1}$$

hence

$$\Phi S = \mu S - \delta \cdot mS > 0.$$

By Theorem 3 in Chapter 7, §11 (Hahn decomposition), there is a Φ-*positive* set $P \in \mathcal{M}$ with a Φ-negative complement $-P = S - P \in \mathcal{M}$.

Clearly, $mP > 0$; for if $mP = 0$, the m-continuity of μ would imply $\mu P = 0$, hence

$$\Phi P = \mu P - \delta \cdot mP = 0,$$

contrary to $\Phi P \geq \Phi S > 0$.

Also, $P \supseteq Y$ and $Y \in \mathcal{M}$ implies $\Phi Y \geq 0$; so by (1),

$$0 \leq \mu Y - \delta \cdot mY.$$

Taking $Y = X \cap P$, we get

$$\delta \cdot m(X \cap P) \leq \mu(X \cap P) \leq \mu X,$$

as required. $\square$

Lemma 2. *With m, μ, and S as in Lemma 1, let $\mathcal{H}$ be the set of all maps $g\colon S \to E^*$, $\mathcal{M}$-measurable and nonnegative on S, such that*

$$\int_X g\,dm \leq \mu X$$

for every set X from $\mathcal{M}$.

Then there is $f \in \mathcal{H}$ with

$$\int_S f\,dm = \max_{g \in \mathcal{H}} \int_S g\,dm.$$

Proof. $\mathcal{H}$ is not empty; e.g., $g = 0$ is in $\mathcal{H}$. We now show that

$$(2) \qquad (\forall\, g, h \in \mathcal{H}) \quad g \vee h = \max(g, h) \in \mathcal{H}.$$

Indeed, $g \vee h$ is ≥ 0 and $\mathcal{M}$-measurable on S, as g and h are.

Now, given $X \in \mathcal{M}$, let $Y = X(g > h)$ and $Z = X(g \leq h)$. Dropping "dm" for brevity, we have

$$\int_X (g \vee h) = \int_Y (g \vee h) + \int_Z (g \vee h) = \int_Y g + \int_Z h \leq \mu Y + \mu Z = \mu X,$$

proving (2).

Let

$$k = \sup_{g \in \mathcal{H}} \int_S g\,dm \in E^*.$$

Proceeding as in Problem 13 of Chapter 7, §6, and using (2), one easily finds a sequence $\{g_n\}\!\uparrow$, $g_n \in \mathcal{H}$, such that

$$\lim_{n \to \infty} \int_S g_n\,dm = k.$$

(Verify!) Set

$$f = \lim_{n \to \infty} g_n.$$

(It exists since $\{g_n\}\!\uparrow$.) By Theorem 4 in §6,

$$k = \lim_{n \to \infty} \int_S g_n = \int_S f.$$

Also, f is $\mathcal{M}$-measurable and ≥ 0 on S, as all g_n are; and if $X \in \mathcal{M}$, then

$$(\forall\, n) \quad \int_X g_n \leq \mu X;$$

hence

$$\int_X f = \lim_{n\to\infty} \int_X g_n \leq \mu X.$$

Thus $f \in \mathcal{H}$ and

$$\int_S f = k = \sup_{g \in H} \int_S g,$$

i.e.,

$$\int_S f = \max_{g \in \mathcal{H}} \int_S g \leq \mu S < \infty.$$

This completes the proof. $\square$

Note 1. As $\mu < \infty$ and $f \geq 0$, Corollary 1 in §5 shows that f can be made *finite* on all of S. Also, f is m-integrable on S.

Theorem 1 (Radon–Nikodym). *If $(S, \mathcal{M}, m)$ is a σ-finite measure space, if $S \in \mathcal{M}$, and if*

$$\mu\colon \mathcal{M} \to E^n\,(C^n)$$

is a generalized m-continuous *measure, then*

$$\mu = \int f\, dm \ \text{ on } \mathcal{M}$$

for at least one map

$$f\colon S \to E^n\,(C^n),$$

$\mathcal{M}$-measurable on S.

Moreover, if h is another such map, then $mS(f \neq h) = 0$

The last part of Theorem 1 means that f is "*essentially unique*." We call f the *Radon–Nikodym (RN) derivative of* μ, with respect to m.

Proof. Via components (Theorem 5 in Chapter 7, §11), all reduces to the case

$$\mu\colon \mathcal{M} \to E^1.$$

Then Theorem 4 (Jordan decomposition) in Chapter 7, §11, yields

$$\mu = \mu^+ - \mu^-,$$

where μ^+ and μ^- are finite *measures* (≥ 0), both m-*continuous* (Corollary 3 from Chapter 7, §11). Therefore, all reduces to the case $0 \leq \mu < \infty$.

Suppose first that m, too, is finite. Then if $\mu = 0$, just take $f = 0$.

If, however, $\mu S > 0$, take $f \in \mathcal{H}$ as in Lemma 2 and Note 1; f is nonnegative, bounded, and $\mathcal{M}$-measurable on S,

$$\int f \leq \mu < \infty,$$

and

$$\int_S f\,dm = k = \sup_{g \in \mathcal{H}} \int_S g\,dm.$$

We claim that f is the required map.

Indeed, let

$$\nu = \mu - \int f\,dm;$$

so ν is a finite m-continuous *measure* (≥ 0) on $\mathcal{M}$. (Why?) We must show that $\nu = 0$.

Seeking a contradiction, suppose $\nu S > 0$. Then by Lemma 1, there are $P \in \mathcal{M}$ and $\delta > 0$ such that $mP > 0$ and

$$(\forall\, X \in \mathcal{M}) \quad \nu X \geq \delta \cdot m(X \cap P).$$

Now let

$$g = f + \delta \cdot C_P;$$

so g is $\mathcal{M}$-measurable and ≥ 0. Also,

$$\begin{aligned}(\forall\, X \in \mathcal{M}) \quad \int_X g = \int_X f + \delta \int_X C_P &= \int_X f + \delta \cdot m(X \cap P) \\ &\leq \int_X f + \nu(X \cap P) \\ &\leq \int_X f + \nu X = \mu X\end{aligned}$$

by our choice of δ and ν. Thus $g \in \mathcal{H}$. On the other hand,

$$\int_S g = \int_S f + \delta \int_S C_P = k + \delta\, mP > k,$$

contrary to

$$k = \sup_{g \in \mathcal{H}} \int_S g.$$

This proves that $\int f = \mu$, indeed.

Now suppose there is another map $h \in \mathcal{H}$ with

$$\mu = \int h\,dm = \int f\,dm \neq \infty;$$

so

$$\int (f-h)\,dm = 0.$$

(Why?) Let

$$Y = S(f \geq h) \text{ and } Z = S(f < h);$$

so $Y, Z \in \mathcal{M}$ (Theorem 3 of §2) and $f - h$ is *sign-constant* on Y and Z. Also, by construction,

$$\int_Y (f-h)\,dm = 0 = \int_Z (f-h)\,dm.$$

Thus by Theorem 1(h) in §5, $f-h=0$ a.e. on Y, on Z, and hence on $S = Y \cup Z$; that is,

$$mS(f \neq h) = 0.$$

Thus all is proved for the case $mS < \infty$.

Next, let m be *σ-finite*:

$$S = \bigcup_{k=1}^{\infty} S_k \ (\textit{disjoint})$$

for some sets $S_k \in \mathcal{M}$ with $mS_k < \infty$.

By what was shown above, on each S_k there is an $\mathcal{M}$-measurable map $f_k \geq 0$ such that

$$\int_X f_k\,dm = \mu X$$

for all $\mathcal{M}$-sets $X \subseteq S_k$. Fixing such an f_k for each k, define $f\colon S \to E^1$ by

$$f = f_k \quad \text{on } S_k, \quad k = 1, 2, \ldots.$$

Then (Corollary 3 in §1) f is $\mathcal{M}$-measurable and ≥ 0 *on* S.

Taking any $X \in \mathcal{M}$, set $X_k = X \cap S_k$. Then

$$X = \bigcup_{k=1}^{\infty} X_k \ (\textit{disjoint})$$

and $X_k \in \mathcal{M}$. Also,

$$(\forall\, k) \quad \int_{X_k} f\,dm = \int_{X_k} f_k\,dm = \mu X_k.$$

Thus by σ-additivity (Theorem 2 in §5),

$$\int_X f\,dm = \sum_{k=1}^{\infty} \int_{X_k} f\,dm = \sum_k \mu X_k = \mu X < \infty \quad (\mu \text{ is finite!}).$$

Thus f is as required, and its "uniqueness" follows as before. □

Note 2. By Definition 3 in §10, we may write

$$\text{“}d\mu = f\,dm\text{”}$$

for

$$\text{“}\int f\,dm = \mu.\text{”}$$

Note 3. Using Definition 2 in §10 and an easy “componentwise” proof, one shows that Theorem 1 holds also with m replaced by a *generalized* measure s. The formulas

$$\mu = \int f\,dm \text{ and } mS(f \neq h) = 0$$

then are replaced by

$$\mu = \int f\,ds \text{ and } v_s S(f \neq h) = 0.$$

II. Theorem 1 requires μ to be *m-continuous* ($\mu \ll m$). We want to generalize Theorem 1 so as to lift this restriction. First, we introduce a new concept.

Definition.

Given two set functions $s, t\colon \mathcal{M} \to E$ ($\mathcal{M} \subseteq 2^S$), we say that s is *t-singular* ($s \perp t$) iff there is a set $P \in \mathcal{M}$ such that $v_t P = 0$ and

$$(\forall\, X \in \mathcal{M} \mid X \subseteq -P) \quad sX = 0. \tag{3}$$

(We then briefly say “s resides *in* P.”)

For *generalized measures*, this means that

$$(\forall\, X \in \mathcal{M}) \quad sX = s(X \cap P).$$

Why?

Corollary 1. *If the generalized measures $s, u\colon \mathcal{M} \to E$ are t-singular, so is ks for any scalar k (if s is scalar valued, k may be a vector).*

So also are $s \pm u$, provided t is additive.

(Exercise! See Problem 3 below.)

Corollary 2. *If a generalized measure $s\colon \mathcal{M} \to E$ is t-continuous ($s \ll t$) and also t-singular ($s \perp t$), then $s = 0$ on $\mathcal{M}$.*

Proof. As $s \perp t$, formula (3) holds for some $P \in \mathcal{M}$, $v_t P = 0$. Hence for all $X \in \mathcal{M}$,

$$s(X - P) = 0 \text{ (for } X - P \subseteq -P)$$

and

$$v_t(X \cap P) = 0 \text{ (for } X \cap P \subseteq P).$$

As $s \ll t$, we also have $s(X \cap P) = 0$ by Definition 3(i) in Chapter 7, §11. Thus by additivity,

$$sX = s(X \cap P) + s(X - P) = 0,$$

as claimed. $\square$

Theorem 2 (Lebesgue decomposition)**.** *Let $s, t \colon \mathcal{M} \to E$ be generalized measures.*

If v_s is t-finite (*Definition* 3(iii) *in Chapter* 7, §11), *there are generalized measures $s', s'' \colon \mathcal{M} \to E$ such that*

$$s' \ll t \text{ and } s'' \perp t$$

and

$$s = s' + s''.$$

Proof. Let v_0 be the restriction of v_s to

$$\mathcal{M}_o = \{X \in \mathcal{M} \mid v_t X = 0\}.$$

As v_s is a *measure* (Theorem 1 of Chapter 7, §11), so is v_0 (for $\mathcal{M}_0$ is a σ-ring; verify!).

Thus by Problem 13 in Chapter 7, §6, we fix $P \in \mathcal{M}_0$, with

$$v_s P = v_0 P = \max\{v_s X \mid X \in \mathcal{M}_0\}.$$

As $P \in \mathcal{M}_0$, we have $v_t P = 0$; hence

$$|sP| \le v_s P < \infty$$

(for v_s is *t-finite*).

Now define s', s'', v', and v'' by setting, for each $X \in \mathcal{M}$,

$$s'X = s(X - P); \tag{4}$$

$$s''X = s(X \cap P); \tag{5}$$

$$v'X = v_s(X - P); \tag{6}$$

$$v''X = v_s(X \cap P). \tag{7}$$

As s and v_s are σ-additive, so are s', s'', v', and v''. (Verify!) Thus $s', s'' \colon \mathcal{M} \to E$ are generalized measures, while v' and v'' are *measures* (≥ 0).

We have

$$(\forall X \in \mathcal{M}) \quad sX = s(X - P) + s(X \cap P) = s'X + s''X;$$

i.e.,

$$s = s' + s''.$$

Similarly one obtains $v_s = v' + v''$.

Also, by (5), since $X \cap P = \emptyset$,

$$-P \supseteq X \text{ and } X \in \mathcal{M} \Longrightarrow s''X = 0,$$

while $v_t P = 0$ (see above). Thus s'' *is t-singular*, residing in P.

To prove $s' \ll t$, it suffices to show that $v' \ll t$ (for by (4) and (6), $v'X = 0$ *implies* $|s'X| = 0$).

Assume the opposite. Then

$$(\exists\, Y \in \mathcal{M}) \quad v_t Y = 0$$

(i.e., $Y \in \mathcal{M}_0$), but

$$0 < v'Y = v_s(Y - P).$$

So by additivity,

$$v_s(Y \cup P) = v_s P + v_s(Y - P) > v_s P,$$

with $Y \cup P \in \mathcal{M}_0$, *contrary to*

$$v_s P = \max\{v_s X \mid X \in \mathcal{M}_0\}.$$

This contradiction completes the proof. □

Note 4. The set function s'' in Theorem 2 is *bounded* on $\mathcal{M}$. Indeed, $s'' \perp t$ yields a set $P \in \mathcal{M}$ such that

$$(\forall\, X \in \mathcal{M}) \quad s''(X - P) = 0;$$

and $v_t P = 0$ implies $v_s P < \infty$. (Why?) Hence

$$s''X = s''(X \cap P) + s''(X - P) = s''(X \cap P).$$

As $s = s' + s''$, we have

$$|s''| \le |s| + |s'| \le v_s + v_{s'};$$

so

$$|s''X| = |s''(X \cap P)| \le v_s P + v_{s'} P.$$

But $v_{s'} P = 0$ by t-continuity (Theorem 2 of Chapter 7, §11). Thus $|s''| \le v_s P < \infty$ on $\mathcal{M}$.

Note 5. The Lebesgue decomposition $s = s' + s''$ in Theorem 2 is *unique*. For if also

$$u' \ll t \text{ and } u'' \perp t$$

and

$$u' + u'' = s = s' + s'',$$

then with P as in Problem 3, $(\forall\, X \in \mathcal{M})$

$$s'(X \cap P) + s''(X \cap P) = u'(X \cap P) + u''(X \cap P) \tag{8}$$

and $v_t(X \cap P) = 0$. But

$$s'(X \cap P) = 0 = u'(X \cap P)$$

by t-continuity; so (8) reduces to

$$s''(X \cap P) = u''(X \cap P),$$

or $s''X = u''X$ (for s'' and u'' *reside* in P). Thus $s'' = u''$ on $\mathcal{M}$.

By Note 4, we may cancel s'' and u'' in

$$s' + s'' = u' + u''$$

to obtain $s' = u'$ also.

Note 6. If $E = E^n$ (C^n), the t-finiteness of v_s in Theorem 2 is redundant, for v_s is even *bounded* (Theorem 6 in Chapter 7, §11).

We now obtain the desired generalization of Theorem 1.

Corollary 3. *If $(S, \mathcal{M}, m)$ is a σ-finite measure space ($S \in \mathcal{M}$), then for any generalized measure*

$$\mu \colon \mathcal{M} \to E^n\,(C^n),$$

*there is a unique m-*singular *generalized measure*

$$s'' \colon \mathcal{M} \to E^n\,(C^n)$$

and a ("essentially" unique) map

$$f \colon S \to E^n\,(C^n),$$

$\mathcal{M}$-measurable and m-integrable on S, with

$$\mu = \int f\,dm + s''.$$

(Note 3 applies here.)

Proof. By Theorem 2 and Note 5, $\mu = s' + s''$ for some (unique) generalized measures $s', s'' \colon \mathcal{M} \to E^n\,(C^n)$, with $s' \ll m$ and $s'' \perp m$.

Now use Theorem 1 to represent s' as $\int f\,dm$, with f as stated. This yields the result. $\square$

Problems on Radon–Nikodym Derivatives and Lebesgue Decomposition

1. Fill in all proof details in Lemma 2 and Theorem 1.

2. Verify the statement following formula (3). Also prove the following:

(i) If $P \in \mathcal{M}$ along with $-P \in \mathcal{M}$, then $s \perp t$ implies $t \perp s$;

(ii) $s \perp t$ iff $v_s \perp t$.

3. Prove Corollary 1.

[Hints: Here $\mathcal{M}$ is a *σ-ring*. Suppose s and u reside in P' and P'', respectively, and $v_t P' = 0 = v_t P''$. Let $P = P' \cup P'' \in \mathcal{M}$. Verify that $v_t P = 0$ (use Problem 8 in Chapter 7, §11). Then show that both s and u reside in P.]

4. Show that if $s\colon \mathcal{M} \to E^*$ is a *signed* measure in $S \in \mathcal{M}$, then $s^+ \perp s^-$ and $s^- \perp s^+$.

5. Fill in all details in the proof of Theorem 2. Also prove the following:

(i) s' and $v_{s'}$ are *absolutely* t-continuous.
[Hint: Use Theorem 2 in Chapter 7, §11.]

(ii) $v_s = v_{s'} + v_{s''}$, $v_{s''} \perp t$.

(iii) If s is a *measure* (≥ 0), so are s' and s''.

6. Verify Note 3 for Theorem 1 and Corollary 3. State and prove both generalized propositions precisely.

*§12. Integration and Differentiation

I. We shall now link RN-derivatives (§11) to those of Chapter 7, §12.

Below, we use the notation of Definition 3 in Chapter 7, §10 and Definition 1 of Chapter 7, §12. (Review them!) In particular,

$$m\colon \mathcal{M}^* \to E^*$$

is Lebesgue measure in E^n (presupposed in such terms as "a.e.," etc.); s is an *arbitrary* set function. For convenience, we set

$$s'(\bar{p}) = 0$$

and

$$\int_X f\, dm = 0,$$

unless defined otherwise; thus s' and $\int_X f$ exist *always*.

We start with several lemmas that go back to Lebesgue.

Lemma 1. *With the notation of Definition 3 of Chapter 7, §10, the functions*

$$\overline{D}s,\ \underline{D}s,\ \text{and } s'$$

are Lebesgue measurable on E^n for any set function

$$s\colon \mathcal{M}' \to E^* \quad (\mathcal{M}' \supseteq \overline{\mathcal{K}}).$$

Proof. By definition,

$$\overline{D}s(\bar{p}) = \inf_r h_r(\bar{p}),$$

where

$$h_r(\bar{p}) = \sup\left\{\frac{sI}{mI} \;\middle|\; I \in \mathcal{K}_{\bar{p}}^r\right\}$$

and

$$\mathcal{K}_{\bar{p}}^r = \left\{I \in \overline{\mathcal{K}} \;\middle|\; \bar{p} \in I,\ dI < \frac{1}{r}\right\}, \quad r = 1, 2, \ldots .$$

As is easily seen (verify!),

$$E^n(h_r > a) = \bigcup\left\{I \in \overline{\mathcal{K}} \;\middle|\; a < \frac{sI}{mI},\ dI < \frac{1}{r}\right\}, \quad a \in E^*. \tag{1}$$

The right-side union is Lebesgue measurable by Problem 2 in Chapter 7, §10. Thus by Theorem 1 of §2, the function h_r is measurable on E^n for $r = 1, 2, \ldots$, and so is

$$\overline{D}s = \inf_r h_r$$

by Lemma 1 of §2 and Definition 3 in Chapter 7, §10. Similarly for $\underline{D}s$.

Hence by Corollary 1 in §2, the set

$$A = E^n(\underline{D}s = \overline{D}s)$$

is measurable. As $s' = \overline{D}s$ on A, s' is measurable on A and also on $-A$ (by convention, $s' = 0$ on $-A$), hence on all of E^n. $\square$

Lemma 2. *With the same notation, let $s\colon \mathcal{M}' \to E^*$ ($\mathcal{M}' \supseteq \overline{\mathcal{K}}$) be a regular measure in E^n. Let $A \in \mathcal{M}^*$ and $B \in \mathcal{M}'$ with $A \subseteq B$, and $a \in E^1$.*

If

$$\overline{D}s > a \quad \text{on } A,$$

then

$$a \cdot mA \le sB.$$

Proof. Fix $\varepsilon > 0$. By regularity (Definition 4 in Chapter 7, §7), there is an open set $G \supseteq B$, with

$$sB + \varepsilon \ge sG.$$

Now let

$$\mathcal{K}^\varepsilon = \left\{I \in \overline{\mathcal{K}} \;\middle|\; I \subseteq G,\ sI \ge (a - \varepsilon)\, mI\right\}.$$

As $\overline{D}s > a$, the definition of $\overline{D}s$ implies that $\mathcal{K}^\varepsilon$ is a Vitali covering of A. (Verify!)

Thus Theorem 1 in Chapter 7, §10, yields a *disjoint* sequence $\{I_k\} \subseteq \mathcal{K}^\varepsilon$, with

$$m\Big(A - \bigcup_k I_k\Big) = 0$$

and

$$mA \le m(A - \bigcup I_k) + m\bigcup I_k = 0 + m\bigcup I_k = \sum_k mI_k.$$

As

$$\bigcup I_k \subseteq G \text{ and } sB + \varepsilon \ge sG$$

(by our choice of $\mathcal{K}^\varepsilon$ and G), we obtain

$$sB + \varepsilon \ge s\bigcup_k I_k = \sum_k sI_k \ge (a - \varepsilon)\sum_k mI_k \ge (a - \varepsilon)\, mA.$$

Thus

$$(a - \varepsilon)\, mA \le sB + \varepsilon.$$

Making $\varepsilon \to 0$, we obtain the result. □

Lemma 3. *If*

$$t = s \pm u,$$

with $s, t, u\colon \mathcal{M}' \to E^*$ and $\mathcal{M}' \supseteq \overline{\mathcal{K}}$, and if u is differentiable at a point $\bar{p} \in E^n$, then

$$\overline{D}t = \overline{D}s \pm u' \textit{ and } \underline{D}t = \underline{D}s \pm u' \textit{ at } \bar{p}.$$

The proof, from definitions, is left to the reader (Chapter 7, §12, Problem 7).

Lemma 4. *Any m-continuous measure $s\colon \mathcal{M}^* \to E^1$ is strongly regular.*

Proof. By Corollary 3 of Chapter 7, §11, $v_s = s < \infty$ (s is finite!). Thus v_s is certainly m-finite.

Hence by Theorem 2 in Chapter 7, §11, s is *absolutely* m-continuous. So given $\varepsilon > 0$, there is $\delta > 0$ such that

$$(\forall X \in \mathcal{M}^* \mid mX < \delta) \quad sX < \varepsilon.$$

Now, let $A \in \mathcal{M}^*$. By the strong regularity of Lebesgue measure m (Chapter 7, §8, Theorem 3(b)), there is an open set $G \supseteq A$ and a closed $F \subseteq A$ such that

$$m(A - F) < \delta \text{ and } m(G - A) < \delta.$$

Thus by our choice of δ,

$$s(A - F) < \varepsilon \text{ and } s(G - A) < \varepsilon,$$

as required. □

Lemma 5. *Let s, s_k $(k = 1, 2, \dots)$ be finite m-continuous measures, with $s_k \nearrow s$ or $s_k \searrow s$ on $\mathcal{M}^*$.*

If the s_k are a.e. differentiable, then

$$\overline{D}s = \underline{D}s = \lim_{k\to\infty} s_k' \text{ a.e.}$$

Proof. Let first $s_k \nearrow s$. Set

$$t_k = s - s_k.$$

By Corollary 2 in Chapter 7, §11, all t_k are m-continuous, hence strongly regular (Lemma 4). Also, $t_k \searrow 0$ (since $s_k \nearrow s$). Hence

$$t_k I \geq t_{k+1} I \geq 0$$

for each cube I; and the definition of $\overline{D}t_k$ implies that

$$\overline{D}t_k \geq \overline{D}t_{k+1} \geq \underline{D}t_{k+1} \geq 0.$$

As $\{\overline{D}t_k\}\!\downarrow$, $\lim_{k\to\infty} \overline{D}t_k$ *exists* (pointwise). Now set

$$A_r = E^n\Big(\lim_{k\to\infty} \overline{D}t_k \geq \frac{1}{r}\Big), \quad r = 1, 2, \dots.$$

By Lemma 1 (and Lemma 1 in §2), $A_r \in \mathcal{M}^*$. Since

$$\overline{D}t_k \geq \lim_{i\to\infty} \overline{D}t_i \geq \frac{1}{r}$$

on A_r, Lemma 2 yields

$$\frac{1}{r}\, mA_r \leq t_k A_r.$$

As $t_k \searrow 0$, we have

$$\frac{1}{r}\, mA_r \leq \lim_{k\to\infty} t_k A_r = 0.$$

Thus

$$mA_r = 0, \quad r = 1, 2, \dots.$$

Also, as is easily seen,

$$E^n\Big(\lim_{k\to\infty} \overline{D}t_k > 0\Big) = \bigcup_{r=1}^{\infty} E^n\Big(\lim_{k\to\infty} \overline{D}t_k \geq \frac{1}{r}\Big) = \bigcup_{r=1}^{\infty} A_r$$

and

$$m \bigcup_{r=1}^{\infty} A_r = 0.$$

Hence

$$\lim_{k\to\infty} \overline{D}t_k \leq 0 \quad \text{a.e.}$$

As

$$\overline{D}t_k \geq \underline{D}t_k \geq 0$$

(see above), we get

$$\lim_{k\to\infty} \overline{D}t_k = 0 = \lim_{k\to\infty} \underline{D}t_k \quad \text{a.e. on } E^n.$$

Now, as $t_k = s - s_k$ and as the s_k are differentiable, Lemma 3 yields

$$\overline{D}t_k = \overline{D}s - s_k' \text{ and } \underline{D}t_k = \underline{D}s - s_k' \quad \text{a.e.}$$

Thus

$$\lim_{k\to\infty} (\overline{D}s - s_k') = 0 = \lim(\underline{D}s - s_k'),$$

i.e.,

$$\overline{D}s = \lim_{k\to\infty} s_k' = \underline{D}s \quad \text{a.e.}$$

This settles the case $s_k \nearrow s$.

In the case $s_k \searrow s$, one only has to set $t_k = s_k - s$ and proceed as before. (Verify!) □

Lemma 6. *Given $A \in \mathcal{M}^*$, $mA < \infty$, let*

$$s = \int C_A \, dm$$

on $\mathcal{M}^*$. Then s is a.e. differentiable, and

$$s' = C_A \text{ a.e. on } E^n.$$

(C_A = characteristic function of A.)

Proof.[1] First, let A be *open* and let $\bar{p} \in A$.

Then A contains some $G_{\bar{p}}(\delta)$ and hence also all cubes $I \in \overline{\mathcal{K}}$ with $dI < \delta$ and $\bar{p} \in I$.

Thus for such $I \in \overline{\mathcal{K}}$,

$$sI = \int_I C_A \, dm = \int_I (1) \, dm = mI;$$

i.e.,

$$\frac{sI}{mI} = 1 = C_A(\bar{p}), \quad \bar{p} \in A.$$

Hence by Definition 1 of Chapter 7, §12,

$$s'(\bar{p}) = 1 = C_A(\bar{p})$$

if $\bar{p} \in A$; i.e., $s' = C_A$ on A.

---

[1] Differentiability follows by Theorem 4 of Chapter 7, §12, but we obtain it anyway.

We claim that

$$\overline{D}s = s' = 0 \quad \text{a.e. } on\ -A.$$

To prove it, note that

$$s = \int C_A \, dm$$

is a finite (why?) m-continuous measure on $\mathcal{M}^*$. By Lemma 4, s is strongly regular. Also, as $sI \geq 0$ for any $I \in \overline{\mathcal{K}}$, we certainly have

$$\overline{D}s \geq \underline{D}s \geq 0.$$

(Why?) Now let

$$B = E^n(\overline{D}s > 0) = \bigcup_{r=1}^{\infty} B_r, \tag{2}$$

where

$$B_r = E^n\left(\overline{D}s \geq \frac{1}{r}\right), \quad r = 1, 2, \ldots. \tag{3}$$

We have to show that $m(B - A) = 0$.

Suppose

$$m(B - A) > 0.$$

Then by (2), we must have $m(B_r - A) > 0$ for at least one B_r; we fix this B_r. Also, by (3),

$$\overline{D}s \geq \frac{1}{r} \text{ on } B_r - A$$

(even on *all* of B_r). Thus by Lemma 2,

$$0 < \frac{1}{r} m(B_r - A) \leq s(B_r - A) = \int_{B_r - A} C_A \, dm. \tag{4}$$

But this is impossible. Indeed, as $C_A = 0$ on $-A$ (hence on $B_r - A$), the integral in (4) cannot be > 0. This refutes the assumption $m(B - A) > 0$; so by (2),

$$m(E^n(\overline{D}s > 0) - A) = 0;$$

i.e.,

$$\overline{D}s = 0 = \underline{D}s \quad \text{a.e. on } -A.$$

We see that

$$s' = 0 = C_A \quad \text{a.e. on } -A,$$

and

$$s' = 1 = C_A \quad \text{on } A,$$

proving the lemma for *open* sets A.

Now take *any* $A \in \mathcal{M}^*$, $mA < \infty$. As Lebesgue measure is regular (Chapter 7, §8, Theorem 3(b)), we find for each $k \in N$ an open set $G_k \supseteq A$, with

$$m(G_k - A) < \frac{1}{k} \text{ and } G_k \supseteq G_{k+1}.$$

Let

$$s_k = \int C_{G_k}\, dm.$$

Then $s_k \searrow s$ on $\mathcal{M}^*$ (see Problem 5(ii) in §6). Also, by what was shown above, the s_k are differentiable, with $s_k' = C_{G_k}$ a.e.

Hence by Lemma 5,

$$\overline{D}s = \underline{D}s = \lim_{k\to\infty} C_{G_k} = C_A \text{ (a.e.)}.$$

The lemma is proved. $\square$

Theorem 1. *Let* $f\colon E^n \to E^*$ (E^r, C^r) *be* m*-integrable, at least on each cube in* E^n*. Then the set function*

$$s = \int f\, dm$$

is differentiable, with $s' = f$*, a.e. on* E^n*.*[2]

Thus s' is the *RN-derivative* of s with respect to Lebesgue measure m (Theorem 1 in §11).

Proof. As E^n is a *countable* union of cubes (Lemma 2 in Chapter 7, §2), it suffices to show that $s' = f$ a.e. on each open cube J, with s differentiable a.e. on J.

Thus fix such a $J \neq \emptyset$ and restrict s and m to

$$\mathcal{M}_0 = \{X \in \mathcal{M}^* \mid X \subseteq J\}.$$

This does not affect s' on J; for as J is *open*, any sequence of cubes

$$I_k \to \bar{p} \in J$$

terminates *inside* J anyway.

When so restricted,

$$s = \int f$$

is a generalized measure in J; for $\mathcal{M}_0$ is a σ-ring (verify!), and f is integrable on J. Also, m is strongly regular, and s is m-continuous.

---

[2] Recall that $\int f$ is *always* defined by our convention.

First, suppose f is $\mathcal{M}_0$-*simple* on J, say,

$$f = \sum_{i=1}^{q} a_i C_{A_i},$$

say, with $0 < a_i < \infty$, $A_i \in \mathcal{M}^*$, and

$$J = \bigcup_{i=1}^{q} A_i \ (\textit{disjoint}).$$

Then

$$s = \int f = \sum_{i=1}^{q} a_i \int C_{A_i}.$$

Hence by Lemma 6 above and by Theorem 1 in Chapter 7, §12, s is differentiable a.e. (as each $\int C_{A_i}$ is), and

$$s' = \sum_{i=1}^{q} a_i \Big(\int C_{A_i} \Big)' = \sum_{i=1}^{q} a_i \, C_{A_i} = f \ \text{(a.e.)},$$

as required.

The general case reduces (via components and the formula $f = f^+ - f^-$) to the case $f \geq 0$, with f measurable (even integrable) on J.

By Problem 6 in §2, then, we have $f_k \nearrow f$ for some simple maps $f_k \geq 0$. Let

$$s_k = \int f_k \text{ on } M_0, \ k = 1, 2, \ldots .$$

Then all s_k and $s = \int f$ are finite *measures* and $s_k \nearrow s$, by Theorem 4 in §6. Also, by what was shown above, each s_k is differentiable a.e. on J, with $s'_k = f_k$ (a.e.). Thus as in Lemma 5,

$$\overline{D}s = \underline{D}s = s' = \lim_{k \to \infty} s'_k = \lim f_k = f \ \text{(a.e.) on } J,$$

with $s' = f \neq \pm\infty$ (a.e.), as f is integrable on J. Thus all is proved. $\square$

II. So far we have considered *Lebesgue* ($\overline{\mathcal{K}}$) differentiation. However, our results easily extend to Ω-differentiation (Definition 2 in Chapter 7, §12).

The proof is even simpler. Thus in Lemma 1, the union in formula (1) is *countable* (as $\overline{\mathcal{K}}$ is replaced by the countable set family Ω); hence it is μ-measurable. In Lemma 2, the use of the Vitali theorem is replaced by Theorem 3 in Chapter 7, §12. Otherwise, one only has to replace Lebesgue measure m by μ on $\mathcal{M}$. Once the lemmas are established (reread the proofs!), we obtain the following.

Theorem 2. *Let S, ρ, Ω, and $\mu\colon \mathcal{M} \to E^*$ be as in Definition 2 of Chapter 7, §12. Let $f\colon S \to E^*\,(E^r, C^r)$ be μ-integrable on each $A \in \mathcal{M}$ with $\mu A < \infty$.*

Then the set function

$$s = \int f\, d\mu$$

is Ω-differentiable, with $s' = f$, (a.e.) on S.

Proof. Recall that S is a countable union of sets $U_n^i \in \Omega$ with $0 < \mu U_n^i < \infty$. As μ^* is $\mathcal{G}$-regular, each U_n^i lies in an *open* set $J_n^i \in \mathcal{M}$ with

$$\mu J_n^i < \mu U_n^i + \varepsilon_n^i < \infty.$$

Also, f is μ-measurable (even integrable) on J_n^i. Dropping a null set, assume that f is $\mathcal{M}$-measurable on $J = J_n^i$.

From here, proceed exactly as in Theorem 1, replacing m by μ. □

Both theorems combined yield the following result.

Corollary 1. *If $s\colon \mathcal{M}' \to E^*\,(E^r, C^r)$ is an m-continuous and m-finite generalized measure in E^n, then s is $\overline{\mathcal{K}}$-differentiable a.e. on E^n, and $ds = s'\, dm$ (see Definition 3 in §10) in any $A \in \mathcal{M}^*$ $(mA < \infty)$.*[3]

Similarly for Ω-differentiation.

Proof. Given $A \in \mathcal{M}^*$ $(mA < \infty)$, there is an open set $J \supseteq A$ such that

$$mJ < mA + \varepsilon < \infty.$$

As before, restrict s and m to

$$\mathcal{M}_0 = \{X \in \mathcal{M}^* \mid X \subseteq J\}.$$

Then by assumption, s is finite and m-continuous on $\mathcal{M}_0$ (a σ-ring); so by Theorem 1 in §11,

$$s = \int f\, dm$$

on $\mathcal{M}_0$ for some m-integrable map f on J.

Hence by our present Theorem 1, s is differentiable, with $s' = f$ a.e. on J, and so

$$s = \int f = \int s' \text{ on } \mathcal{M}_0.$$

This implies $ds = s'\, dm$ in A.

For Ω-differentiation, use Theorem 2. □

---

[3] The restriction $mA < \infty$ is redundant if s is *finite*.

Corollary 2 (change of measure)**.** *Let s be as in Corollary* 1*. Subject to Note* 1 *in* §10*, if f is s-integrable on $A \in \mathcal{M}^*$ $(mA < \infty)$,*[4] *then fs' is m-integrable on A and*

$$\int_A f\,ds = \int_A fs'\,dm.$$

Similarly for Ω-derivatives, with m replaced by μ.

Proof. By Corollary 1, $ds = s'\,dm$ in A. Thus Theorem 6 of §10 yields the result. □

Note 1. In particular, Corollary 2 applies to m-continuous signed LS measures $s = s_\alpha$ in E^1 (see end of §11). If $A = [a, b]$, then s_α is surely finite on s_α-measurable subsets of A; so Corollaries 1 and 2 show that

$$\int_A f\,ds_\alpha = \int_A fs'_\alpha\,dm = \int_A f\alpha'\,dm,$$

since $s'_\alpha = \alpha'$. (See Problem 9 in Chapter 7, §12.)

Note 2. Moreover, $s = s_\alpha$ (see Note 1) is absolutely m-continuous iff α is absolutely continuous in the stronger sense (Problem 2 in Chapter 4, §8).

Indeed, assuming the latter, fix $\varepsilon > 0$ and choose δ as in Definition 3 of Chapter 7, §11. Then if $mX < \delta$, we have

$$X \subseteq \bigcup I_k \text{ (\textit{disjoint})}$$

for some intervals $I_k = (a_k, b_k]$, with

$$\delta > \sum mI_k = \sum (b_k - a_k).$$

Hence

$$|sX| \le \sum |sI_k| < \varepsilon.$$

(Why?) Similarly for the converse.[5]

Problems on Differentiation and Related Topics

1. Fill in all proof details in this section. Verify footnote 4 and Note 2.

2. Given a measure $s\colon \mathcal{M}' \to E^*$ $(\mathcal{M}' \supseteq \overline{\mathcal{K}})$, prove that

(i) s is topological;

(ii) its Borel restriction σ is strongly regular; and

(iii) $\underline{D}s$, $\overline{D}s$, and s' do not change if s or m are restricted to the Borel field $\mathcal{B}$ in E^n; neither does this affect the propositions on $\overline{\mathcal{K}}$-differentiation proved here.

---

[4] The restriction $mA < \infty$ is redundant if s is *finite*.

[5] Note that $s\{a\} = 0$ if s is m-continuous.

[Hints: (i) Use Lemma 2 of Chapter 7, §2. (ii) Use also Problem 10 in Chapter 7, §7. (iii) All depends *on* $\overline{\mathcal{K}}$.]

3. What analogues to 2(i)–(iii) apply to Ω-differentiation in E^n? In (S, ρ)?

4. (i) Show that any m-singular measure s in E^n, finite on $\overline{\mathcal{K}}$, has a *zero derivative* (a.e.).

(ii) For Ω-derivatives, prove that this holds if s is also *regular*.

[Hint for (i): By Problem 2, we may assume s regular (if not, replace it by σ). Suppose

$$mE^n(\overline{D}s > 0) > a > 0$$

and find a contradiction to Lemma 2.]

5. Give another proof for Theorem 4 in Chapter 7, §12.

[Hint: Fix an open cube $J \in \overline{\mathcal{K}}$. By Problem 2(iii), restrict s and m to

$$\mathcal{M}_0 = \{X \in \mathcal{B} \mid X \subseteq J\}$$

to make them *finite*. Apply Corollary 2 in §11 to s. Then use Problem 4, Theorem 1 of the present section, and Theorem 1 of Chapter 7, §12.

For Ω-differentiation, assume s *regular*; argue as in Corollary 1, using Corollary 2 of §11.]

6. Prove that if

$$F(x) = L\int_a^x f\,dm \quad (a \le x \le b),^6$$

with $f\colon E^1 \to E^*$ $(E^n,\ C^n)$ m-integrable on $A = [a, b]$, then F is differentiable, with $F' = f$, a.e. on A.

[Hint: Via components, reduce all to the case $f \ge 0$, $F\!\uparrow$ on A.

Let

$$s = \int f\,dm$$

on $\mathcal{M}^*$. Let $t = m_F$ be the F-induced LS measure. Show that $s = t$ on intervals in A; so $s' = t' = F'$ a.e. on A (Problem 9 in Chapter 7, §11). Use Theorem 1.]

---

[6] Here $L\int_a^x f\,dm = \int_{[a,x]} f\,dm$; m =Lebesgue measure.

Chapter 9

Calculus Using Lebesgue Theory

§1. L-Integrals and Antiderivatives

I. Lebesgue theory makes it possible to strengthen many calculus theorems. We shall start with functions *on* E^1, $f\colon E^1 \to E$. (A reader who has omitted the "starred" part of Chapter 8, §7, will have to set $E = E^*$ (E^n, C^n) throughout.)

By *L-integrals* of such functions, we mean integrals with respect to *Lebesgue* measure m in E^1. Notation:

$$L\int_a^b f = L\int_a^b f(x)\,dx = L\int_{[a,b]} f$$

and

$$L\int_b^a f = -L\int_a^b f.$$

For *Riemann* integrals, we replace "L" by "R." We compare such integrals with *antiderivatives* (Chapter 5, §5), denoted

$$\int_a^b f,$$

without the "L" or "R." Note that

$$L\int_{[a,b]} f = L\int_{(a,b)} f,$$

etc., since $m\{a\} = m\{b\} = 0$ here.

Theorem 1. *Let $f\colon E^1 \to E$ be L-integrable on $A = [a,b]$. Set*

$$H(x) = L\int_a^x f, \quad x \in A.$$

Then the following are true.

(i) *The function f is the derivative of H at any $p \in A$ at which f is finite and continuous.* (*At a and b, continuity and derivatives may be one-sided from within.*)

(ii) *The function H is absolutely continuous on A;*[1] *hence $V_H[A] < \infty$.*[2,3]

Proof. (i) Let $p \in (a, b]$, $q = f(p) \neq \pm\infty$. Let f be *left* continuous at p; so, given $\varepsilon > 0$, we can fix $c \in (a, p)$ such that

$$|f(x) - q| < \varepsilon \text{ for } x \in (c, p).$$

Then

$$(\forall\, x \in (c,p)) \quad \left| L \int_x^p (f - q) \right| \leq L \int_x^p |f - q|$$
$$\leq L \int_x^p (\varepsilon) = \varepsilon \cdot m[x,p] = \varepsilon\,(p - x).$$

But

$$L \int_x^p (f - q) = L \int_x^p f - L \int_x^p q,$$
$$L \int_x^p q = q\,(p - x), \quad \text{and}$$
$$L \int_x^p f = L \int_a^p f - L \int_a^x f$$
$$= H(p) - H(x).$$

Thus

$$|H(p) - H(x) - q(p - x)| \leq \varepsilon\,(p - x);$$

i.e.,

$$\left| \frac{H(p) - H(x)}{p - x} - q \right| \leq \varepsilon \quad (c < x < p).$$

Hence

$$f(p) = q = \lim_{x \to p^-} \frac{\Delta H}{\Delta x} = H'_-(p).$$

If f is *right* continuous at $p \in [a, b)$, a similar formula results for $H'_+(p)$. This proves clause (i).

---

[1] This is true even in the *stronger* sense, as in Problem 2 of Chapter 5, §8, or in §2 (*next* to this).

[2] Recall that $V_H[A]$ is the *total variation* of H on A (Chapter 5, §§7–8).

[3] Part (ii) is true even if f is not L-integrable, only $L \int_a^b |f| < \infty$ is needed.

(ii) Let $\varepsilon > 0$ be given. Then Theorem 6 in Chapter 8, §6, yields a $\delta > 0$ such that

$$\left| L \int_X f \right| \le L \int_X |f| < \varepsilon \tag{1}$$

whenever

$$mX < \delta \text{ and } A \supseteq X, \ X \in \mathcal{M}.$$

Here we may set

$$X = \bigcup_{i=1}^{r} A_i \ (\textit{disjoint})$$

for some intervals

$$A_i = (a_i, b_i) \subseteq A$$

so that

$$mX = \sum_i mA_i = \sum_i (b_i - a_i) < \delta.$$

Then (1) implies that

$$\varepsilon > L \int_X |f| = \sum_i L \int_{A_i} |f| \ge \sum_i \left| L \int_{a_i}^{b_i} f \right| = \sum_i |H(b_i) - H(a_i)|.$$

Thus

$$\sum_i |H(b_i) - H(a_i)| < \varepsilon$$

whenever

$$\sum_i (b_i - a_i) < \delta$$

and

$$A \supseteq \bigcup_i (a_i, b_i) \ (\textit{disjoint}).$$

(This is what we call "*absolute continuity in the stronger sense*.") By Problem 2 in Chapter 5, §8, this implies "absolute continuity" in the sense of Chapter 5, §8, hence $V_H[A] < \infty$. $\square$

Note 1. The converse to (i) *fails*: the differentiability of H at p does not imply the continuity of its derivative f at p (Problem 6 in Chapter 5, §2).

Note 2. If f is continuous on $A - Q$ (Q *countable*), Theorem 1 shows that H is a *primitive* (*antiderivative*): $H = \int f$ on A.[4] Recall that "Q countable" implies $mQ = 0$, but not conversely. Observe that we may always assume $a, b \in Q$.

---

[4] See Definition 1 in Chapter 5, §5.

We can now prove a generalized version of the so-called *fundamental theorem of calculus*, widely used for computing integrals via antiderivatives.

Theorem 2. *If* $f\colon E^1 \to E$ *has a primitive* F *on* $A = [a, b]$*, and if* f *is bounded on* $A - P$ *for some* P *with* $mP = 0$*, then* f *is L-integrable on* A*, and*

$$L\int_a^x f = F(x) - F(a) \quad \textit{for all } x \in A. \tag{2}$$

Proof. By Definition 1 of Chapter 5, §5, F is relatively continuous and finite on $A = [a, b]$, hence *bounded* on A (Theorem 2 in Chapter 4, §8).

It is also differentiable, with $F' = f$, on $A - Q$ for a *countable* set $Q \subseteq A$, with $a, b \in Q$. We fix this Q along with P.

As we deal with A only, we surely may *redefine* F and f *on* $-A$:

$$F(x) = \begin{cases} F(a) & \text{if } x < a, \\ F(b) & \text{if } x > b, \end{cases}$$

and $f = 0$ on $-A$. Then f is bounded *on* $-P$, while F is bounded and continuous *on* E^1, and $F' = f$ *on* $-Q$; so $F = \int f$ *on* E^1.[5]

Also, for $n = 1, 2, \ldots$ and $t \in E^1$, set

$$f_n(t) = n\Big[F\Big(t + \frac{1}{n}\Big) - F(t)\Big] = \frac{F(t + 1/n) - F(t)}{1/n}. \tag{3}$$

Then

$$f_n \to F' = f \quad \text{on } -Q;$$

i.e., $f_n \to f$ (*a.e.*) on E^1 (as $mQ = 0$).

By (3), each f_n is bounded and continuous (as F is). Thus by Theorem 1 of Chapter 8, §3, F and all f_n are m-measurable on A (even on E^1). So is f by Corollary 1 of Chapter 8, §3.

Moreover, by boundedness, F and f_n are L-integrable on finite intervals. *So is* f. For example, let

$$|f| \le K < \infty \text{ on } A - P;$$

as $mP = 0$,

$$\int_A |f| \le \int_A (K) = K \cdot mA < \infty,$$

proving integrability. Now, as

$$F = \int f \text{ on any interval } \Big[t, t + \frac{1}{n}\Big],$$

---

[5] See Definition 1 from Chapter 5, §5.

Corollary 1 in Chapter 5, §4 yields

$$(\forall\, t \in E^1) \quad \left|F\Big(t+\frac{1}{n}\Big) - F(t)\right| \le \sup_{t \in -Q} |F'(t)|\,\frac{1}{n} \le \frac{K}{n}.$$

Hence

$$|f_n(t)| = n\left|F\Big(t+\frac{1}{n}\Big) - F(t)\right| \le K;$$

i.e., $|f_n| \le K$ for all n.

Thus f and f_n satisfy Theorem 5 of Chapter 8, §6, with $g = K$. By Note 1 there,

$$\lim_{n\to\infty} L\int_a^x f_n = L\int_a^x f.$$

In the next lemma, we show that *also*

$$\lim_{n\to\infty} L\int_a^x f_n = F(x) - F(a),$$

which will complete the proof. $\square$

Lemma 1. *Given a finite continuous* $F\colon E^1 \to E$ *and given* f_n *as in* (3), *we have*

$$\lim_{n\to\infty} L\int_a^x f_n = F(x) - F(a) \quad \textit{for all } x \in E^1. \tag{4}$$

Proof. As before, F and f_n are bounded, continuous, and *L-integrable* on any $[a, x]$ or $[x, a]$. Fixing a, let

$$H(x) = L\int_a^x F, \quad x \in E^1.$$

By Theorem 1 and Note 2, $H = \int F$ *also in the sense of Chapter* 5, §5, with $F = H'$ (derivative of H) on E^1.

Hence by Definition 2 the same section,

$$\int_a^x F = H(x) - H(a) = H(x) - 0 = L\int_a^x F;$$

i.e.,

$$L\int_a^x F = \int_a^x F,$$

and so

$$\begin{aligned} L\int_a^x f_n(t)\,dt &= n\int_a^x F\Big(t+\frac{1}{n}\Big)\,dt - n\int_a^x F(t)\,dt \\ &= n\int_{a+1/n}^{b+1/n} F(t)\,dt - n\int_a^x F(t)\,dt. \end{aligned}$$

(We computed

$$\int F(t+1/n)\,dt$$

by Theorem 2 in Chapter 5, §5, with $g(t)=t+1/n$.) Thus by additivity,

$$L\int_a^x f_n = n\int_{a+1/n}^{x+1/n} F - n\int_a^x F = n\int_x^{x+1/n} F - n\int_a^{a+1/n} F. \tag{5}$$

But

$$n\int_x^{x+1/n} F = \frac{H(x+\frac{1}{n})-H(x)}{\frac{1}{n}} \to H'(x)=F(x).$$

Similarly,

$$\lim_{n\to\infty} n\int_a^{a+1/n} F = F(a).$$

This combined with (5) proves (4), and hence Theorem 2, too. □

We also have the following corollary.

Corollary 1. *If* $f\colon E^1 \to E^*$ $(E^n,\ C^n)$ *is R-integrable on* $A=[a,b]$, *then*

$$(\forall\, x\in A)\quad R\int_a^x f = L\int_a^b f = F(x)-F(a), \tag{6}$$

provided F *is primitive to* f *on* A.[6]

This follows from Theorem 2 by Definition (c) and Theorem 2 of Chapter 8, §9.

Caution. Formulas (2) and (6) may fail if f is unbounded, or if F is not a primitive *in the sense of Definition* 1 *of Chapter* 5, §5: We need $F'=f$ on $A-Q$, Q *countable* ($mQ=0$ is not enough!). Even R-integrability (which makes f bounded and a.e. continuous) does not suffice if

$$F \neq \int f.$$

For examples, see Problems 2–5.

Corollary 2. *If* f *is relatively continuous and finite on* $A=[a,b]$ *and has a bounded derivative on* $A-Q$ (Q *countable*), *then* f' *is L-integrable on* A *and*

$$L\int_a^x f' = f(x)-f(a) \quad \textit{for } x\in A. \tag{7}$$

This is simply Theorem 2 with F, f, P replaced by f, f', Q, respectively.

---

[6] We assumed that $E=E^*$ $(E^n,\ C^n)$ since R-integrals were defined for *that* case only.

Corollary 3. *If in Theorem 2 the primitive*

$$F = \int f$$

is exact *on some $B \subseteq A$, then*

$$f(x) = \frac{d}{dx}\, L \int_a^x f, \quad x \in B. \tag{8}$$

(Recall that $\frac{d}{dx} F(x)$ is classical notation for $F'(x)$.)

Proof. By (2), this holds on $B \subseteq A$ if $F' = f$ there. $\square$

II. Note that under the assumptions of Theorem 2,

$$L \int_a^x f = F(x) - F(a) = \int_a^x f.$$

Thus all laws governing the *primitive* $\int f$ apply to $L \int f$. For example, Theorem 2 of Chapter 5, §5, yields the following corollary.

Corollary 4 (change of variable)**.** *Let $g\colon E^1 \to E^1$ be relatively continuous on $A = [a, b]$ and have a bounded derivative on $A - Q$ (Q countable).*

Suppose that $f\colon E^1 \to E$ (real or not) has a primitive on $g[A]$, exact on $g[A - Q]$, and that f is bounded on $g[A - Q]$.

Then f is L-integrable on $g[A]$, the function

$$(f \circ g)\, g'$$

is L-integrable on A, and

$$L \int_a^b f(g(x))\, g'(x)\, dx = L \int_p^q f(y)\, dy, \tag{9}$$

where $p = g(a)$ and $q = g(b)$.

For this and other applications of primitives, see Problem 9. However, often a *direct* approach is *stronger* (though not simpler), as we illustrate next.

Lemma 2 (Bonnet)**.** *Suppose $f\colon E^1 \to E^1$ is ≥ 0 and monotonically decreasing on $A = [a, b]$. Then, if $g\colon E^1 \to E^1$ is L-integrable on A, so also is fg, and*

$$L \int_a^b fg = f(a) \cdot L \int_a^c g \quad \textit{for some } c \in A. \tag{10}$$

Proof. The L-integrability of fg follows by Theorem 3 in Chapter 8, §6, as f is monotone and bounded, hence even *R-integrable* (Corollary 3 in Chapter 8, §9).

Using this and Lemma 1 of the same section, fix for each n a $\mathcal{C}$-partition

$$\mathcal{P}_n = \{A_{ni}\} \quad (i = 1, 2, \ldots, q_n)$$

of A so that

$$(\forall\, n) \quad \frac{1}{n} > \overline{S}(f, \mathcal{P}_n) - \underline{S}(f, \mathcal{P}_n) = \sum_{i=1}^{q_n} w_{ni}\, mA_{ni}, \tag{11}$$

where we have set

$$w_{ni} = \sup f[A_{ni}] - \inf f[A_{ni}].$$

Consider any such $\mathcal{P} = \{A_i\}$, $i = 1, \ldots, q$ (we drop the "n" for brevity). If $A_i = [a_{i-1}, a_i]$, then since $f{\downarrow}$,

$$w_i = f(a_{i-1}) - f(a_i) \geq |f(x) - f(a_{i-1})|, \quad x \in A_i.$$

Under *Lebesgue* measure (Problem 8 of Chapter 8, §9), we may set

$$A_i = [a_{i-1}, a_i] \quad (\forall\, i)$$

and still get

$$\begin{aligned} L\int_A fg = {} & \sum_{i=1}^{q} f(a_{i-1}) L\int_{A_i} g(x)\, dx \\ & + \sum_{i=1}^{q} L\int_{A_i} [f(x) - f(a_{i-1})]\, g(x)\, dx. \end{aligned} \tag{12}$$

(Verify!) Here $a_0 = a$ and $a_q = b$.

Now, set

$$G(x) = L\int_a^x g$$

and rewrite the *first* sum (call it r or r_n) as

$$\begin{aligned} r &= \sum_{i=1}^{q} f(a_{i-1})\, [G(a_i) - G(a_{i-1})] \\ &= \sum_{i=1}^{q-1} G(a_i)\, [f(a_{i-1}) - f(a_i)] + G(b)\, f(a_{q-1}), \end{aligned}$$

or

$$r = \sum_{i=1}^{q-1} G(a_i)\, w_i + G(b)\, f(a_{q-1}), \tag{13}$$

because $f(a_{i-1}) - f(a_i) = w_i$ and $G(a) = 0$.

Now, by Theorem 1 (with H, f replaced by G, g), G is continuous on $A = [a, b]$; so G attains a *largest* value K and a *least* value k on A.

As $f{\downarrow}$ and $f \geq 0$ on A, we have

$$w_i \geq 0 \text{ and } f(a_{q-1}) \geq 0.$$

Thus, replacing $G(b)$ and $G(a_i)$ by K (or k) in (13) and noting that

$$\sum_{i=1}^{q-1} w_i = f(a) - f(a_{q-1}),$$

we obtain

$$kf(a) \leq r \leq Kf(a);$$

more fully, with $k = \min G[A]$ and $K = \max G[A]$,

$$(\forall\, n) \quad kf(a) \leq r_n \leq Kf(a). \tag{14}$$

Next, let s (or rather s_n) be the *second* sum in (12). Noting that

$$w_i \geq |f(x) - f(a_{i-1})|,$$

suppose first that $|g| \leq B$ (*bounded*) on A.

Then for all n,

$$|s_n| \leq \sum_{i=1}^{q_n} L\int_{A_{ni}} (w_{ni}\, B) = B \sum_{i=1}^{q_n} w_{ni}\, mA_{ni} < \frac{B}{n} \to 0 \quad \text{(by (11))}.$$

But by (12),

$$L\int_A fg = r_n + s_n \quad (\forall\, n).$$

As $s_n \to 0$,

$$L\int_A fg = \lim_{n\to\infty} r_n,$$

and so by (14),

$$kf(a) \leq L\int_A fg \leq Kf(a).$$

By continuity, $f(a)G(x)$ takes on the intermediate value $L\int_A fg$ at some $c \in A$; so

$$L\int_A fg = f(a)\, G(c) = f(a) L\int_a^c g,$$

since

$$G(x) = L\int_a^x f.$$

Thus all is proved for a *bounded* g.

The passage to an unbounded g is achieved by the so-called *truncation* method described in Problems 12 and 13. (Verify!) $\square$

Corollary 5 (second law of the mean)**.** *Let* $f\colon E^1 \to E^1$ *be monotone on* $A = [a, b]$*. Then if* $g\colon E^1 \to E^1$ *is L-integrable on* A*, so also is* fg*, and*

$$L\int_a^b fg = f(a)\,L\int_a^c g + f(b)\,L\int_c^b g \quad \textit{for some } c \in A. \tag{15}$$

Proof. If, say, $f\downarrow$ on A, set

$$h(x) = f(x) - f(b).$$

Then $h \geq 0$ and $h\downarrow$ on A; so by Lemma 2,

$$\int_a^b gh = h(a)\,L\int_a^c g \quad \text{for some } c \in A.$$

As

$$h(a) = f(a) - f(b),$$

this easily implies (15).

If $f\uparrow$, apply this result to $-f$ to obtain (15) again. □

Note 3. We may restate (15) as

$$(\exists\, c \in A) \quad L\int_a^b fg = p\,L\int_a^c g + q\,L\int_c^b g,$$

provided either

(i) $f\uparrow$ and $p \leq f(a+) \leq f(b-) \leq q$, or

(ii) $f\downarrow$ and $p \geq f(a+) \geq f(b-) \geq q$.

This statement slightly strengthens (15).

To prove clause (i), *redefine*

$$f(a) = p \text{ and } f(b) = q.$$

Then still $f\uparrow$; so (15) applies and yields the desired result. Similarly for (ii). For a *continuous* g, see also Problem 13(ii′) in Chapter 8, §9, based on *Stieltjes* theory.

III. We now give a useful analogue to the notion of a primitive.

Definition.

A map $F\colon E^1 \to E$ is called an *L-primitive* or an *indefinite L-integral* of $f\colon E^1 \to E$, on $A = [a, b]$ iff f is L-integrable on A and

$$F(x) = c + L\int_a^x f \tag{16}$$

for all $x \in A$ and some fixed finite $c \in E$.

Notation:

$$F = L\int f \quad \left(\textit{not } F = \int f\right)$$

or

$$F(x) = L\int f(x)\,dx \quad \text{on } A.$$

By (16), *all L-primitives of f on A differ by finite constants only.*

If $E = E^*\,(E^n,\,C^n)$, one can use this concept to *lift the boundedness restriction on f* in Theorem 2 and the corollaries of this section. The proof will be given in §2. However, for comparison, we state the main theorems already now.

***Theorem 3.** *Let*

$$F = L\int f \quad \textit{on } A = [a, b]$$

for some $f\colon E^1 \to E^*\,(E^n,\,C^n)$.

Then F is differentiable, with

$$F' = f \quad \textit{a.e. on } A.$$

In classical notation,

$$f(x) = \frac{d}{dx}\, L\int_a^x f(t)\,dt \quad \textit{for almost all } x \in A. \tag{17}$$

A proof was sketched in Problem 6 of Chapter 8, §12. (It is brief but requires more "starred" material than used in §2.)

***Theorem 4.** *Let $F\colon E^1 \to E^n\,(C^n)$ be differentiable on $A = [a, b]$ (at a and b differentiability may be one sided). Let $F' = f$ be L-integrable on A.*

Then

$$L\int_a^x f = F(x) - F(a) \quad \textit{for all } x \in A. \tag{18}$$

Problems on L-Integrals and Antiderivatives

1. Fill in proof details in Theorems 1 and 2, Lemma 1, and Corollaries 1–3.

1′. Verify Note 2.

2. Let F be Cantor's function (Problem 6 in Chapter 4, §5). Let

$$G = \bigcup_{k,i} G_{ki}$$

(G_{ki} as in that problem). So $[0, 1] - G = P$ (Cantor's set); $mP = 0$ (Problem 10 in Chapter 7, §8).

Show that F is differentiable ($F' = 0$) *on* G. By Theorems 2 and 3 of Chapter 8, §9,

$$R\int_0^1 F' = L\int_0^1 F' = L\int_G F' = 0$$

exists, yet $F(1) - F(0) = 1 - 0 \neq 0$.

Does this contradict Corollary 1? Is F a genuine antiderivative of f? If not, find one.

3. Let

$$F = \begin{cases} 0 & \text{on } [0, \frac{1}{2}), \text{ and} \\ 1 & \text{on } [\frac{1}{2}, 1]. \end{cases}$$

Show that

$$R\int_0^1 F' = 0$$

exists, yet

$$F(1) - F(0) = 1 - 0 = 1.$$

What is wrong?

[Hint: A *genuine* primitive of F' (call it ϕ) has to be *relatively continuous* on $[0, 1]$; find ϕ and show that $\phi(1) - \phi(0) = 0$.]

4. What is wrong with the following computations?

(i) $L\int_{-1}^{\frac{1}{2}} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{\frac{1}{2}} = -1.$

(ii) $L\int_{-1}^{1} \frac{dx}{x} = \ln|x|\Big|_{-1}^{1} = 0.$ Is there a primitive on the *whole* interval?

[Hint: See hint to Problem 3.]

(iii) How about $L\int_{-1}^{1} \frac{|x|}{x}\,dx$ (cf. examples (a) and (b) of Chapter 5, §5)?

5. Let

$$F(x) = x^2 \cos\frac{\pi}{x^2}, \quad F(0) = 1.$$

Prove the following:

(i) F is differentiable on $A = [0, 1]$.

(ii) $f = F'$ is bounded on any $[a, b] \subset (0, 1)$, but not on A.

(iii) Let

$$a_n = \sqrt{\frac{2}{4n+1}} \text{ and } b_n = \frac{1}{\sqrt{2n}} \text{ for } n = 1, 2, \ldots .$$

Show that

$$A \supseteq \bigcup_{n=1}^{\infty} [a_n, b_n] \ (\textit{disjoint})$$

and

$$L \int_{a_n}^{b_n} f = \frac{1}{2n};$$

so

$$L \int_a^b f \geq L \int_{\bigcup_{n=1}^{\infty} [a_n, b_n]} f \geq \sum_{n=1}^{\infty} \frac{1}{2n} = \infty,$$

and $f = F'$ is *not* L-integrable on A.

What is wrong? Is there a contradiction to Theorem 2?

6. Consider both

(a) $f(x) = \dfrac{\sin x}{x}$, $f(0) = 1$, and

(b) $f(x) = \dfrac{1 - e^{-x}}{x}$, $f(0) = 1$.

In each case, show that f is continuous on $A = [0, 1]$ and

$$R \int_A f \leq 1$$

exists, yet it does not "work out" via primitives. What is wrong? Does a primitive exist?

To use Corollary 1, first expand $\sin x$ and e^{-x} in a Taylor series and find the series for

$$\int f$$

by Theorem 3 of Chapter 5, §9.

Find

$$R \int_A f$$

approximately, to within $1/10$, using the remainder term of the series to estimate accuracy.

[Hint: Primitives *exist*, by Theorem 2 of Chapter 5, §11, even though they are none of the known "calculus functions."]

7. Take A, $G_n = (a_n, b_n)$, and P $(mP > 0)$ as in Problem 17(iii) of Chapter 7, §8.

Define $F = 0$ on P and

$$F(x) = (x - a_n)^2(x - b_n)^2 \sin \frac{1}{(b_n - a_n)(x - a_n)(x - b_n)} \quad \text{if } x \notin P.$$

Prove that F has a *bounded* derivative f, yet f is *not* R-integrable on A; so Theorem 2 applies, but Corollary 1 does not.
[Hints: If $p \notin P$, compute $F'(p)$ as in calculus.

If $p \in P$ and $x \to p+$ *over* $A - P$, then x is always in some (a_n, b_n), $p \leq a_n < x$. (Why?) Deduce that $\Delta x = x - p > x - a_n$ and

$$\left|\frac{\Delta F}{\Delta x}\right| \leq (x - a_n)(b - a)^2 \leq |\Delta x|(b - a)^2;$$

so $F'_+(p) = 0$. (What if $x \to p+$ *over* P?) Similarly, show that $F'_- = 0$ *on* P.

Prove however that $F'(x)$ oscillates from 1 to -1 as $x \to a_n+$ or $x \to b_n-$, hence also as $x \to p \in P$ (why?); so F' is discontinuous on all of P, *with* $mP > 0$. Now use Theorem 3 in Chapter 8, §9.]

⇒**8.** If

$$Q \subseteq A = [a, b]$$

and $mQ = 0$, find a continuous map $g\colon A \to E^1$, $g \geq 0$, $g\uparrow$, with

$$g' = +\infty \quad \text{on } Q.$$

[Hints: By Theorem 2 of Chapter 7, §8, fix $(\forall\, n)$ an *open* $G_n \supseteq Q$, with

$$mG_n < 2^{-n}.$$

Set

$$g_n(x) = m(G_n \cap [a, x])$$

and

$$g = \sum_{n=1}^{\infty} g_n$$

on A; $\sum g_n$ converges *uniformly* on A. (Why?)

By Problem 4 in Chapter 7, §9, and Theorem 2 of Chapter 7, §4, each g_n (hence g) is continuous. (Why?) If $[p, x] \subseteq G_n$, show that

$$g_n(x) = g_n(p) + (x - p),$$

so

$$\frac{\Delta g_n}{\Delta x} = 1$$

and

$$\frac{\Delta g}{\Delta x} = \sum_{n=1}^{\infty} \frac{\Delta g_n}{\Delta x} \to \infty.]$$

9. (i) Prove Corollary 4.

(ii) State and prove earlier analogues for Corollary 5 of Chapter 5, §5, and Theorems 3 and 4 from Chapter 5, §10.

[Hint for (i): For *primitives*, this is Problem 3 in Chapter 5, §5. As $g[Q]$ is *countable* (Problem 2 in Chapter 1, §9) and f is bounded on

$$g[A] - g[Q] \subseteq g[A - Q],$$

f satisfies Theorem 2 on $g[A]$, with $P = g[Q]$, while $(f \circ g)\, g'$ satisfies it *on* A.]

⇒**10.** Show that if $h\colon E^1 \to E^*$ is L-integrable on $A = [a, b]$, and

$$(\forall\, x \in A) \quad L\int_a^x h = 0,$$

then $h = 0$ a.e. on A.

[Hints: Let $K = A(h > 0)$ and $H = A - K$, with, say, $mK = \varepsilon > 0$.

Then by Corollary 1 in Chapter 7, §1 and Definition 2 of Chapter 7, §5,

$$H \subseteq \bigcup_n B_n \ (\textit{disjoint})$$

for some *intervals* $B_n \subseteq A$, with

$$\sum_n mB_n < mH + \varepsilon = mH + mK = mA.$$

(Why?) Set $B = \bigcup_n B_n$; so

$$\int_B h = \sum_n \int_{B_n} h = 0$$

(for $L \int h = 0$ on *intervals* B_n). Thus

$$\int_{A-B} h = \int_A h - \int_B h = 0.$$

But $B \supseteq H$; so

$$A - B \subseteq A - H = K,$$

where $h > 0$, even though $m(A - B) > 0$. (Why?)

Hence find a contradiction to Theorem 1(h) of Chapter 8, §5. Similarly, disprove that $mA(h < 0) = \varepsilon > 0$.]

⇒**11.** Let $F\!\uparrow$ on $A = [a, b]$, $|F| < \infty$, with derived function $F' = f$. Taking Theorem 3 from Chapter 7, §10, for granted, prove that

$$L\int_a^x f \le F(x) - F(a), \quad x \in A.$$

[Hints: With f_n as in (3), F and f_n are bounded on A and measurable by Theorem 1 of Chapter 8, §2. (Why?) Deduce that $f_n \to f$ (a.e.) on A. Argue as in Lemma 1 using *Fatou's* lemma (Chapter 8, §6, Lemma 2).]

12. ("Truncation.") Prove that if $g\colon S \to E$ is m-integrable on $A \in \mathcal{M}$ in a measure space $(S, \mathcal{M}, m)$, then for any $\varepsilon > 0$, there is a *bounded*, $\mathcal{M}$-measurable and integrable on A map $g_0\colon S \to E$ such that

$$\int_A |g - g_0|\, dm < \varepsilon.$$

[Outline: Redefine $g = 0$ on a null set, to make g $\mathcal{M}$-*measurable on* A. Then for $n = 1, 2, \ldots$ set

$$g_n = \begin{cases} g & \text{on } A(|g| < n), \text{ and} \\ 0 & \text{elsewhere.} \end{cases}$$

(The function g_n is called the nth *truncate* of g.)

Each g_n is bounded and $\mathcal{M}$-measurable on A (why?), and

$$\int_A |g|\, dm < \infty$$

by integrability. Also, $|g_n| \le |g|$ and $g_n \to g$ (*pointwise*) on A. (Why?)

Now use Theorem 5 from Chapter 8, §6, to show that one of the g_n may serve as the desired g_o.]

13. Fill in all proof details in Lemma 2. Prove it for *unbounded* g.

[Hints: By Problem 12, fix a *bounded* g_o ($|g_o| \le B$), with

$$L \int_A |g - g_o| < \frac{1}{2} \frac{\varepsilon}{f(a) - f(b)}.$$

Verify that

$$\begin{aligned} |s_n| \le \sum_{i=1}^{q_n} \int_{A_{ni}} w_{ni}|g| &\le \sum_i \int_{A_{ni}} w_{ni}|g_o| + \sum_i \int_{A_{ni}} w_{ni}|g - g_o| \\ &\le B \sum_i w_{ni}\, mA_{ni} + \sum_i \int_{A_{ni}} [f(a) - f(b)]\, |g - g_o| \\ &< \frac{1}{n} + \int_A [f(a) - f(b)]\, |g - g_o| < \frac{1}{n} + \frac{1}{2}\varepsilon. \end{aligned}$$

For all $n > 2/\varepsilon$, we get $|s_n| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$. Hence $s_n \to 0$. Now finish as in the text.]

14. Show that Theorem 4 fails if F is not differentiable at some $p \in A$.

[Hint: See Problems 2 and 3.]

§2. More on L-Integrals and Absolute Continuity

I. In this section, we presuppose the "starred" §10 in Chapter 7. First, however, we add some new ideas that do not require any starred material. The notation is as in §1.

Definition 1.

Given $F\colon E^1 \to E$, $p \in E^1$, and $q \in E$, we write

$$q \sim DF(p)$$

and call q an *F-derivate* at p iff

$$q = \lim_{k\to\infty} \frac{F(x_k) - F(p)}{x_k - p}$$

for *at least one* sequence $x_k \to p$ $(x_k \neq p)$.[1]

If F has a *derivative* at p, it is the *only* F-derivate at p; otherwise, there may be many derivates at p (finite or not).

Such derivates *must* exist if $E = E^1$ (E^*). Indeed, given any $p \in E^1$, let

$$x_k = p + \frac{1}{k} \to p;$$

let

$$y_k = \frac{F(x_k) - F(p)}{x_k - p}, \quad k = 1, 2, \ldots$$

By the compactness of E^* (Chapter 4, §6, example (d)), $\{y_k\}$ must have a subsequence $\{y_{k_i}\}$ *with a limit* $q \in E^*$ (e.g., take $q = \underline{\lim}\, y_k$), and so $q \sim DF(p)$.

We also obtain the following lemma.

Lemma 1. *If $F\colon E^1 \to E^*$ has no* negative *derivates on $A - Q$, where $A = [a, b]$ and $mQ = 0$, and if no derivate of F on A equals $-\infty$, then $F\!\uparrow$ on A.*

Proof. First, suppose F has no negative derivates on A at all. Fix $\varepsilon > 0$ and set

$$G(x) = F(x) + \varepsilon x.$$

Seeking a contradiction, suppose $a \leq p < q \leq b$, yet $G(q) < G(p)$. Then if

$$r = \frac{1}{2}(p + q),$$

one of the intervals $[p, r]$ and $[r, q]$ (call it $[p_1, q_1]$) satisfies $G(q_1) < G(p_1)$.

Let

$$r_1 = \frac{1}{2}(p_1 + q_1).$$

Again, one of $[p_1, r_1]$ and $[r_1, q_1]$ (call it $[p_2, q_2]$) satisfies $G(q_2) < G(p_2)$. Let

$$r_2 = \frac{1}{2}(p_2 + q_2),$$

and so on.

---

[1] "$DF(p)$" stands for *"an F-derivate at p."*

Thus obtain contracting intervals $[p_n, q_n]$, with

$$G(q_n) < G(p_n), \quad n = 1, 2, \ldots .$$

Now, by Theorem 5 of Chapter 4, §6, let

$$p_o \in \bigcap_{n=1}^{\infty} [p_n, q_n].$$

Then set $x_n = q_n$ if $G(q_n) < G(p_o)$, and $x_n = p_n$ otherwise. Then

$$\frac{G(x_n) - G(p_o)}{x_n - p_o} < 0$$

and $x_n \to p_o$. By the compactness of E^*, fix a subsequence

$$\frac{G(x_{n_k}) - G(p_o)}{x_{n_k} - p_o} \to c \in E^*,$$

say. Then $c \leq 0$ is a G-derivate at $p_o \in A$.

But this is impossible; for by our choice of G and our assumption, all derivates of G are > 0. (Why?)

This contradiction shows that $a \leq p < q \leq b$ implies $G(p) \leq G(q)$, i.e.,

$$F(p) + \varepsilon p \leq F(q) + \varepsilon q.$$

Making $\varepsilon \to 0$, we obtain $F(p) \leq F(q)$ when $a \leq p < q \leq b$, i.e., $F\!\uparrow$ on A.

Now, for the general case, let Q be the set of all $p \in A$ that have at least one $DF(p) < 0$; so $mQ = 0$.

Let g be as in Problem 8 of §1; so $g' = \infty$ on Q. Given $\varepsilon > 0$, set

$$G = F + \varepsilon g.$$

As $g\!\uparrow$, we have

$$(\forall\, x, p \in A) \quad \frac{G(x) - G(p)}{x - p} \geq \frac{F(x) - F(p)}{x - p}.$$

Hence $DG(p) \geq 0$ if $p \notin Q$.

If, however, $p \in Q$, then $g'(p) = \infty$ implies $DG(p) \geq 0$. (Why?) Thus *all* $DG(p)$ are ≥ 0; so by what was proved above, $G\!\uparrow$ on A. It follows, as before, that $F\!\uparrow$ on A, also. The lemma is proved. $\square$

We now proceed to prove Theorems 3 and 4 of §1. To do this, we shall need only one "starred" theorem (Theorem 3 of Chapter 7, §10).

Proof of Theorem 3 of §1. (1) First, let f be *bounded*:

$$|f| \leq K \quad \text{on } A.$$

Via components and by Corollary 1 of Chapter 8, §6, all reduces to the *real positive case* $f \geq 0$ on A. (Explain!)

Then (Theorem 1(f) of Chapter 8, §5) $a \leq x < y \leq b$ implies

$$L\int_a^x f \leq L\int_a^y f,$$

i.e., $F(x) \leq F(y)$; so $F\uparrow$ and $F' \geq 0$ on A.

Now, by Theorem 3 of Chapter 7, §10, F is a.e. differentiable on A. Thus exactly as in Theorem 2 in §1, we set

$$f_n(t) = \frac{F(t+\frac{1}{n}) - F(t)}{\frac{1}{n}} \to F'(t) \text{ a.e.}$$

Since all f_n are m-measurable on A (why?), so is F'. Moreover, as $|f| \leq K$, we obtain (as in Lemma 1 of §1)

$$|f_n(x)| = n\left(L\int_x^{x+1/n} f\right) \leq n \cdot \frac{K}{n} = K.$$

Thus by Theorem 5 from Chapter 8, §6 (with $g = K$),

$$L\int_a^x F' = \lim_{n\to\infty} L\int_a^x f_n = L\int_a^x f$$

(Lemma 1 of §1). Hence

$$L\int_a^x (F' - f) = 0, \quad x \in A,$$

and so (Problem 10 in §1) $F' = f$ (a.e.) as claimed.

(2) If f is not bounded, we still can reduce all to the case $f \geq 0$, $f\colon E^1 \to E^*$, so that $F\uparrow$ and $F' \geq 0$ on A.

If so, we use "truncation": For $n = 1, 2, \ldots,$ set

$$g_n = \begin{cases} f & \text{on } A(f \leq n), \text{ and} \\ 0 & \text{elsewhere.} \end{cases}$$

Then (see Problem 12 in §1) the g_n are L-measurable and *bounded*, hence L-integrable on A, with $g_n \to f$ and

$$0 \leq g_n \leq f$$

on A. By the first part of the proof, then,

$$\frac{d}{dx} L\int_a^x g_n = g_n \quad \text{a.e. on } A, \ n = 1, 2, \ldots.$$

Also, set $(\forall\, n)$

$$F_n(x) = L\int_a^x (f - g_n) \geq 0;$$

so F_n is *monotone* ($\uparrow$) on A. (Why?)

Thus by Theorem 3 in Chapter 7, §10, each F_n has a derivative at almost every $x \in A$,

$$F_n'(x) = \frac{d}{dx}\left(L\int_a^x f - L\int_a^x g_n\right) = F'(x) - g_n(x) \geq 0 \quad \text{a.e. on } A.$$

Making $n \to \infty$ and recalling that $g_n \to f$ on A, we obtain

$$F'(x) - f(x) \geq 0 \quad \text{a.e. on } A.$$

Thus

$$L\int_a^x (F' - f) \geq 0.$$

But as $F\uparrow$ (see above), Problem 11 of §1 yields

$$L\int_a^x F' \leq F(x) - F(a) = L\int_a^x f;$$

so

$$L\int_a^x (F' - f) = L\int_a^x F' - L\int_a^x f \leq 0.$$

Combining, we get

$$(\forall\, x \in A) \quad L\int_a^x (F' - f) = 0;$$

so by Problem 10 of §1, $F' = f$ a.e. on A, as required. $\square$

Proof of Theorem 4 of §1. Via components, all again reduces to a *real* f.[2] Let $(\forall\, n)$

$$g_n = \begin{cases} f & \text{on } A(f \leq n), \\ 0 & \text{on } A(f > n); \end{cases}$$

so $g_n \to f$ (pointwise), $g_n \leq f$, $g_n \leq n$, and $|g_n| \leq |f|$.

This makes each g_n L-integrable on A. Thus as before, by Theorem 5 of Chapter 8, §6,

$$\lim_{n\to\infty} L\int_a^x g_n = L\int_a^x f, \quad x \in A. \tag{1}$$

Now, set

$$F_n(x) = F(x) - L\int_a^x g_n.$$

---

[2] Not $f \geq 0$, though, since Corollary 1 in Chapter 8, §6, does not apply to *differentiation*.

Then by Theorem 3 of §1 (already proved),

$$F_n'(x) = F'(x) - \frac{d}{dx} L\int_a^x g_n = f(x) - g_n(x) \geq 0 \quad \text{a.e. on } A$$

(since $g_n \leq f$).

Thus F_n has solely *nonnegative* derivates on $A - Q$ $(mQ = 0)$. Also, as $g_n \leq n$, we get

$$\frac{1}{x-p} L\int_a^x g_n \leq n,$$

even if $x < p$. (Why?) Hence

$$\frac{\Delta F_n}{\Delta x} \geq \frac{\Delta F}{\Delta x} - n,$$

as

$$F_n(x) = F(x) - L\int_a^x g_n.$$

Thus *none* of the F_n-derivates on A can be $-\infty$.

By Lemma 1, then, F_n is monotone $(\uparrow)$ on A; so $F_n(x) \geq F_n(a)$, i.e.,

$$F(x) - L\int_a^x g_n \geq F(a) - L\int_a^a g_n = F(a),$$

or

$$F(x) - F(a) \geq L\int_a^x g_n, \quad x \in A,\ n = 1, 2, \ldots.$$

Hence by (1),

$$F(x) - F(a) \geq L\int_a^x f, \quad x \in A.$$

For the reverse inequality, apply the same formula to $-f$. Thus we obtain the desired result:

$$F(x) = F(a) + L\int_a^x f \quad \text{for } x \in A. \quad \square \tag{2}$$

Note 1. Formula (2) is equivalent to $F = L\int f$ on A (see the last part of §1). For if (2) holds, then

$$F(x) = c + L\int_a^x f,$$

with $c = F(a)$; so $F = L\int f$ by definition.

Conversely, if

$$F(x) = c + L\int_a^x f,$$

set $x = a$ to find $c = F(a)$.

II. Absolute continuity redefined.

Definition 2.

A map $f\colon E^1 \to E$ is *absolutely continuous* on an interval $I \subseteq E^1$ iff for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\sum_{i=1}^{r}(b_i - a_i) < \delta \text{ implies } \sum_{i=1}^{r}|f(b_i) - f(a_i)| < \varepsilon$$

for any disjoint intervals (a_i, b_i), with $a_i, b_i \in I$.

From now on, this replaces the "weaker" definition given in Chapter 5, §8.

The reader will easily verify the next three "routine" propositions.

Theorem 1. *If $f, g, h\colon E^1 \to E^*\ (C)$ are absolutely continuous on $A = [a, b]$, so are*

$$f \pm g,\ hf,\ \text{and } |f|.$$

So also is f/h if

$$(\exists\, \varepsilon > 0) \quad |h| \geq \varepsilon \text{ on } A.$$

All this also holds if $f, g\colon E^1 \to E$ are vector valued and h is scalar valued. Finally, if $E \subseteq E^*$, then

$$f \vee g,\ f \wedge g,\ f^+,\ \text{and } f^-$$

are absolutely continuous along with f and g.

Corollary 1. *A function $F\colon E^1 \to E^n\ (C^n)$ is absolutely continuous on $A = [a, b]$ iff all its components $F_1, \ldots, F_n$ are.*

Hence a complex function $F\colon E^1 \to C$ is absolutely continuous iff its real and imaginary parts, F_{re} and F_{im}, are.

Corollary 2. *If $f\colon E^1 \to E$ is absolutely continuous on $A = [a, b]$, it is bounded, is uniformly continuous, and has bounded variation, $V_f[a, b] < \infty$, all on A.*

Lemma 2. *If $F\colon E^1 \to E^n\ (C^n)$ is of bounded variation on $A = [a, b]$, then*

(i) *F is a.e. differentiable on A, and*

(ii) *F' is L-integrable on A.*

Proof. Via components (Theorem 4 of Chapter 5, §7), all reduces to the *real* case, $F\colon E^1 \to E^1$.

Then since $V_F[A] < \infty$, we have

$$F = g - h$$

for some *nondecreasing* g and h (Theorem 3 in Chapter 5, §7).

Now, by Theorem 3 from Chapter 7, §10, g and h are a.e. differentiable on A. Hence so is

$$g - h = F.$$

Moreover, $g' \geq 0$ and $h' \geq 0$ since $g\uparrow$ and $h\uparrow$.

Thus for the L-integrability of F', proceed as in Problem 11 in §1, i.e., show that F' is measurable on A and that

$$L\int_a^b F' = L\int_a^b g' - L\int_a^b h'$$

is *finite*. This yields the result. $\square$

Theorem 2 (Lebesgue)**.** *If $F\colon E^1 \to E^n\,(C^n)$ is absolutely continuous on $A = [a, b]$, then the following are true:*

(i*) *F is a.e. differentiable, and F' is L-integrable, on A.*

(ii*) *If, in addition, $F' = 0$ a.e. on A, then F is constant on A.*

Proof. Assertion (i*) is immediate from Lemma 2, since any absolutely continuous function is of bounded variation by Corollary 2.

(ii*) Now let $F' = 0$ a.e. on A. Fix any

$$B = [a, c] \subseteq A$$

and let Z consist of all $p \in B$ at which the *derivative* $F' = 0$.

Given $\varepsilon > 0$, let $\mathcal{K}$ be the set of all closed intervals $[p, x]$, $p < x$, such that

$$\left|\frac{\Delta F}{\Delta x}\right| = \left|\frac{F(x) - F(p)}{x - p}\right| < \varepsilon.$$

By assumption,

$$\lim_{x \to p} \frac{\Delta F}{\Delta x} = 0 \quad (p \in Z),$$

and $m(B - Z) = 0$; $B = [a, c] \in \mathcal{M}^*$. If $p \in Z$, and $x - p$ is small enough, then

$$\left|\frac{\Delta F}{\Delta x}\right| < \varepsilon,$$

i.e., $[p, x] \in \mathcal{K}$.

It easily follows that $\mathcal{K}$ covers Z in the *Vitali* sense (verify!); so for any $\delta > 0$, Theorem 2 of Chapter 7, §10 yields disjoint intervals

$$I_k = [p_k, x_k] \in \mathcal{K},\ I_k \subseteq B,$$

with

$$m^*\left(Z - \bigcup_{k=1}^{q} I_k\right) < \delta,$$

hence also

$$m\left(B - \bigcup_{k=1}^{q} I_k\right) < \delta$$

(for $m(B - Z) = 0$). But

$$\begin{aligned} B - \bigcup_{k=1}^{q} I_k &= [a, c] - \bigcup_{k=1}^{q-1} [p_k, x_k] \\ &= [a, p_1) \cup \bigcup_{k=1}^{q-1} [x_k, p_{k+1}) \cup [x_q, c] \quad (\text{if } x_k < p_k < x_{k+1}); \end{aligned}$$

so

$$m\left(B - \bigcup_{k=1}^{q} I_k\right) = (p_1 - a) + \sum_{k=1}^{q-1} (p_{k+1} - x_k) + (c - x_q) < \delta. \tag{3}$$

Now, as F is absolutely continuous, we can choose $\delta > 0$ so that (3) implies

$$|F(p_1) - F(a)| + \sum_{k=1}^{q-1} |F(p_{k+1}) - F(x_k)| + |F(c) - F(x_q)| < \varepsilon. \tag{4}$$

But $I_k \in \mathcal{K}$ also implies

$$|F(x_k) - F(p_k)| < \varepsilon(x_k - p_k) = \varepsilon \cdot mI_k.$$

Hence

$$\left|\sum_{k=1}^{q} [F(x_k) - F(p_k)]\right| < \varepsilon \sum_{k=1}^{q} mI_k \leq \varepsilon \cdot mB = \varepsilon(c - p).$$

Combining with (4), we get

$$|F(c) - F(a)| \leq \varepsilon(1 + c - a) \to 0 \text{ as } \varepsilon \to 0;$$

so $F(c) = F(a)$. As $c \in A$ was *arbitrary*, F is constant on A, as claimed. □

Note 2. This shows that *Cantor's function* (Problem 6 of Chapter 4, §5) is *not* absolutely continuous, even though it is continuous and monotone, hence of bounded variation on $[0, 1]$. Indeed (see Problem 2 in §1), it has a zero derivative a.e. on $[0, 1]$ but is *not* constant there. Thus absolute continuity, as now defined, *differs* from its "weak" counterpart (Chapter 5, §8).

Theorem 3. *A map* $F\colon E^1 \to E^1\,(C^n)$ *is absolutely continuous on* $A = [a, b]$ *iff*

$$F = L\int f \quad \text{on } A$$

for some function f;[3] *and then*

$$F(x) = F(a) + L\int_a^x f, \quad x \in A.$$

Briefly: *Absolutely continuous maps are exactly all L-primitives.*

Proof. If $F = L\int f$, then by Theorem 1 of §1, F is absolutely continuous on A, and by Note 1,

$$F(x) = F(a) + L\int_a^x f, \quad x \in A.$$

Conversely, if F is absolutely continuous, then by Theorem 2, it is a.e. differentiable and $F' = f$ is L-integrable (all on A). Let

$$H(x) = L\int_a^x f, \quad x \in A.$$

Then H, too, is absolutely continuous and so is $F - H$. Also, by Theorem 3 of §1,

$$H' = f = F',$$

and so

$$(F - H)' = 0 \quad \text{a.e. on } A.$$

By Theorem 2, $F - H = c$; i.e.,

$$F(x) = c + H(x) = c + L\int_a^x f,$$

and so $F = L\int f$ on A, as claimed. $\square$

Corollary 3. *If* $f, F\colon E^1 \to E^*$ (E^n, C^n), *we have*

$$F = L\int f$$

on an interval $I \subseteq E^1$ *iff* F *is absolutely continuous on* I *and* $F' = f$ *a.e. on* I.

(Use Problem 3 in §1 and Theorem 3.)

Note 3. This (or Theorem 3) could serve as a *definition*. Comparing ordinary primitives

$$F = \int f$$

with L-primitives

$$F = L\int f,$$

---

[3] Such as F', the derived function of F.

we see that the former require F to be just *relatively continuous* but allow only a *countable* "exceptional" set Q, while the latter require *absolute* continuity but allow Q to even be uncountable, provided $mQ = 0$.

The simplest and "strongest" kind of absolutely continuous functions are so-called *Lipschitz maps* (see Problem 6). See also Problems 7 and 10.

III. We conclude with another important idea, due to Lebesgue.

Definition 3.

We call $p \in E^1$ a *Lebesgue point* ("*L-point*") of $f\colon E^1 \to E$ iff

(i) f is L-integrable on some $G_p(\delta)$;

(ii) $q = f(p)$ is finite; and

(iii) $\displaystyle\lim_{x\to p} \frac{1}{x-p} L \int_p^x |f-q| = 0.$

The *Lebesgue set* of f consists of all such p.

Corollary 4. *Let*

$$F = L \int f \quad \textit{on } A = [a, b].$$

If $p \in A$ is an L-point of f, then $f(p)$ is the derivative of F at p (but the converse fails).

Proof. By assumption,

$$F(x) = c + L \int_p^x f, \quad x \in G_p(\delta),$$

and

$$\frac{1}{|\Delta x|} \left| L \int_p^x (f-q) \right| \le \frac{1}{|\Delta x|} L \int_p^x |f-q| \to 0$$

as $x \to p$. (Here $q = f(p)$ and $\Delta x = x - p$.)

Thus with $x \to p$, we get

$$\begin{aligned} \left| \frac{F(x)-F(p)}{x-p} - q \right| &= \frac{1}{|x-p|} \left| L \int_p^x f - (x-p)q \right| \\ &= \frac{1}{|x-p|} \left| L \int_p^x f - L \int_p^x (q) \right| \to 0, \end{aligned}$$

as required. □

Corollary 5. *Let $f\colon E^1 \to E^n$ (C^n). Then p is an L-point of f iff it is an L-point for each of the n components, $f_1, \ldots, f_n$, of f.*

(Exercise!)

Theorem 4. *If $f\colon E^1 \to E^*\,(E^n, C^n)$ is L-integrable on $A = [a, b]$, then almost all $p \in A$ are Lebesgue points of f.*

Note that this *strengthens* Theorem 3 of §1.

Proof. By Corollary 5, we need only consider the case $f\colon E^1 \to E^*$.

For any $r \in E^1$, $|f - r|$ is L-integrable on A; so by Theorem 3 of §1, setting

$$F_r(x) = L\int_a^x |f - r|,$$

we get

$$F_r'(p) = \lim_{x \to p} \frac{1}{|x - p|}\, L\int_p^x |f - r| = |f(p) - r| \tag{5}$$

for almost all $p \in A$.

Now, for each r, let A_r be the set of those $p \in A$ for which (5) *fails*; so $mA_r = 0$. Let $\{r_k\}$ be the sequence of all *rationals* in E^1. Let

$$Q = \bigcup_{k=1}^{\infty} A_{r_k} \cup \{a, b\} \cup A_\infty,$$

where

$$A_\infty = A(|f| = \infty);$$

so $mQ = 0$. (Why?)

To finish, we show that *all $p \in A - Q$ are L-points of f.* Indeed, fix any $p \in A - Q$ and any $\varepsilon > 0$. Let $q = f(p)$. Fix a rational r such that

$$|q - r| < \frac{\varepsilon}{3}.$$

Then

$$\big||f - r| - |f - q|\big| \le \big|(f - r) - (f - q)\big| = |q - r| < \frac{\varepsilon}{3} \quad \text{on } A - A_\infty.$$

Hence as $mA_\infty = 0$, we have

$$\left| L\int_p^x |f - r| - L\int_p^x |f - q| \right| \le L\int_p^x \left(\frac{\varepsilon}{3}\right) = \frac{\varepsilon}{3}|x - p|. \tag{6}$$

Since

$$p \notin Q \supseteq \bigcup_k A_{r_k},$$

formula (5) applies. So there is $\delta > 0$ such that $|x - p| < \delta$ implies

$$\left| \left(\frac{1}{|x - p|}\, L\int_p^x |f - r| \right) - |f(p) - r| \right| < \frac{\varepsilon}{3}.$$

As

$$|f(p) - r| = |q - r| < \frac{\varepsilon}{3},$$

we get

$$\begin{aligned}\frac{1}{|x-p|} L\int_p^x |f - r| &\le \left| \left(\frac{1}{|x-p|} L\int_p^x |f - r| \right) - |q - r| \right| + |q - r| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.\end{aligned}$$

Hence

$$L\int_p^x |f - r| < \frac{2\varepsilon}{3}|x - p|.$$

Combining with (6), we have

$$\frac{1}{|x-p|} L\int_p^x |f - q| < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$$

whenever $|x - p| < \delta$. Thus

$$\lim_{x\to p} \frac{1}{|x-p|} L\int_p^x |f - q| = 0,$$

as required. $\square$

Problems on L-Integrals and Absolute Continuity

1. Fill in all details in the proof of Lemma 1 and of Theorems 3 and 4 from §1.

2. Prove Theorem 1 and Corollaries 1, 2, and 5.

2′. Disprove the converse to Corollary 4. (Give an example!)

⇒3. Show that if $F\colon E^1 \to E$ is L-integrable on $A = [a, b]$ and continuous at $p \in A$, then p is an L-point of F.

[Hint: Use the ε, δ definition of continuity.]

4. Complete all proof details for Lemma 2, Theorems 3 and 4, and Corollary 3.

5. Let $F = 1$ on R (= rationals) and $F = 0$ on $E^1 - R$ (Dirichlet function).

Show that F has exactly three derivates (0, $+\infty$, and $-\infty$) at every $p \in E^1$.

⇒6. We say that F is a *Lipschitz map*, or has the uniform *Lipschitz property* on A, iff

$$(\exists\, K \in E^1)\ (\forall\, x, y \in A) \quad |F(x) - F(y)| \le K|x - y|.$$

Prove the following:

(i) Any such F is absolutely continuous on $A = [a, b]$.

(ii) If all derivates of f satisfy

$$|Df(x)| \leq k < \infty, \quad x \in A = [a, b],$$

then f is a Lipschitz map on A.

⇒7. Let $g\colon E^1 \to E^1$ and $f\colon E^1 \to E$ (real or not) be absolutely continuous on $A = [a, b]$ and $g[A]$, respectively.

Prove that $h = f \circ g$ is absolutely continuous on A, *provided that* either f is as in Problem 6, or g is strictly monotone on A.

8. Prove that if $F\colon E^1 \to E^1$ is absolutely continuous on $A = [a, b]$, if $Q \subseteq A$, and if $mQ = 0$, then $m^*F[Q] = 0$ (m = Lebesgue measure).

[Outline: We may assume $Q \subseteq (a, b)$. (Why?)

Fix $\varepsilon > 0$ and take δ as in Definition 2. As m is regular, there is an *open* G,

$$Q \subseteq G \subseteq (a, b),$$

with $mG < \delta$. By Lemma 2 of Chapter 7, §2,

$$G = \bigcup_{k=1}^{\infty} I_k \ (\textit{disjoint})$$

for some $I_k = (a_k, b_k]$.

Let $u_k = \inf F[I_k]$, $v_k = \sup F[I_k]$; so

$$F[I_k] \subseteq [u_k, v_k]$$

and

$$m^*F[I_k] \leq v_k - u_k.$$

Also,

$$\sum (b_k - a_k) = \sum mI_k = mG < \delta.$$

From Definition 2, show that

$$\sum_{k=1}^{\infty} (v_k - u_k) \leq \varepsilon$$

(first consider *partial* sums). As

$$F[Q] \subseteq F[G] \subseteq \bigcup_k F[I_k],$$

get

$$m^*F[Q] \leq \sum_k m^*F[I_k] = \sum_k (v_k - u_k) \leq \varepsilon \to 0.]$$

9. Show that if F is as in Problem 8 and if

$$A = [a, b] \supseteq B,\ B \in \mathcal{M}^*$$

(L-measurable sets), then

$$F[B] \in \mathcal{M}^*.$$

("F preserves $\mathcal{M}^*$-sets.")

[Outline: (i) If B is closed, it is compact, and so is $F[B]$ (Theorems 1 and 4 of Chapter 4, §6).

(ii) If $B \in \mathcal{F}_\sigma$, then

$$B = \bigcup_i B_i, \quad B_i \in \mathcal{F};$$

so by (i),

$$F[B] = \bigcup_i F[B_i] \in \mathcal{F}_\sigma \subseteq \mathcal{M}^*.$$

(iii) If $B \in \mathcal{M}^*$, then by Theorem 2 of Chapter 7, §8,

$$(\exists K \in \mathcal{F}_\sigma) \quad K \subseteq B,\ m(B - K) = 0.$$

Now use Problem 8, with $Q = B - K$.]

⇒**10.** (Change of variable.) Suppose $g\colon E^1 \to E^1$ is absolutely continuous and *one-to-one* on $A = [a, b]$, while $f\colon E^1 \to E^*\,(E^n,\, C^n)$ is L-integrable on $g[A]$.

Prove that $(f \circ g)\, g'$ is L-integrable on A and

$$L\int_a^b (f \circ g)\, g' = L\int_p^q f,$$

where $p = g(a)$ and $q = g(b)$.

[Hints: Let $F = L\int f$ and $H = F \circ g$ on A.

By Theorems 2 and 3 and Problem 7 (end), F and H are absolutely continuous on $g[A]$ and A, respectively; and H' is L-integrable on A. So by Theorem 3,

$$H = L\int H' = L\int (f \circ g) g',$$

as $H' = (f \circ g)\, g'$ a.e. on A.]

11. Setting $f(x) = 0$ if not defined otherwise, find the intervals (if any) on which f is absolutely continuous if $f(x)$ is defined by

(a) $\sin x$;

(b) $\cos 2x$;

(c) $1/x$;

(d) $\tan x$;

(e) x^x;

(f) $x \sin(1/x)$;

(g) $x^2 \sin x^{-2}$ (Problem 5 in §1);

(h) $\sqrt{x^3} \cdot \sin(1/x)$ (verify that $|f'(x)| \leq \frac{3}{2} + x^{-\frac{1}{2}}$).

[Hint: Use Problems 6 and 7.]

§3. Improper (Cauchy) Integrals

Cauchy extended R-integration to *unbounded* sets and functions as follows.

Given $f\colon E^1 \to E$ and assuming that the right-hand side R-integrals and limits exist, define (first for unbounded *sets*, then for unbounded *functions*)

(i) $\displaystyle\int_a^\infty f = \int_{[a,\infty)} f = \lim_{x\to\infty} R\int_a^x f$;

(ii) $\displaystyle\int_{-\infty}^a f = \int_{(-\infty,a]} f = \lim_{x\to-\infty} R\int_x^a f$.

If both

$$\int_0^\infty f \text{ and } \int_{-\infty}^0 f$$

exist, define

$$\int_{-\infty}^\infty f = \int_{(-\infty,0)} f + \int_{[0,\infty)} f.$$

Now, suppose f *is unbounded* near some $p \in A = [a, b]$, i.e., unbounded on

$$A \cap G_{\neg p}$$

for every deleted globe $G_{\neg p}$ about p (such points p are called *singularities*). Then (again assuming existence of the R-integrals and limits), we define

(1) in case of a singularity $p = a$,

$$\int_{a+}^b f = \int_{(a,b]} f = \lim_{x\to a+} R\int_x^b f;$$

(2) if $p = b$, then

$$\int_a^{b-} f = \int_{[a,b)} f = \lim_{x\to b-} R\int_a^x f;$$

(3) if $a < p < b$ and if

$$\int_a^{p-} f \text{ and } \int_{p+}^b f$$

exist, then

$$\int_a^b f = \int_a^{p-} f + \int_p^p f + \int_{p+}^b f.$$

The term

$$\int_p^p f = \int_{[p,p]} f$$

is necessary if *RS- or LS-integrals* are used.[1]

Finally, if A contains *several* singularities, it must be split into subintervals, each with at most *one* endpoint singularity; and $\int_a^b f$ is split accordingly.[2]

We call all such integrals *improper* or *Cauchy* (C) integrals. A C-integral is said to *converge* iff it exists and is *finite*.

This theory is greatly enriched if in the above definitions, one replaces R-integrals by *Lebesgue* integrals, using Lebesgue or LS measure in E^1. (This makes sense even when a Lebesgue integral (proper) does exist; see Theorem 1.) Below, m shall denote such a measure unless stated otherwise.

C-integrals with respect to m will be denoted by

$$C\int_a^\infty f\,dm, \quad C\int_{[a,b)} f, \quad \text{etc.}$$

"Classical" notation:

$$C\int f(x)\,dm(x) \text{ or } C\int f(x)\,dx$$

(the latter if m is *Lebesgue* measure). We omit the "C" if confusion with *proper* integrals $\int_a^x f$ is unlikely.

Note 1. C-integrals are *limits* of integrals, not integrals proper. Yet they may *equal* the latter (Theorem 1 below) and then may be used to *compute* them.

Caution. "Singularities" in $[a, b]$ may affect the *primitive* used in computations (cf. Problem 4 in §1). Then $[a, b]$ must be *split* (see above), and $C\int_a^b f$ splits accordingly. (Additivity applies to C-integrals; see Problem 9, below.)

Examples.

(A) The integral

$$L\int_{-1}^{1/2} \frac{dx}{x^2}$$

---

[1] For RS- and LS-integrals, we may well have $\int_p^p f \neq 0$, $\int_{[a,b]} f \neq \int_{(a,b)} f$, etc.

[2] This also applies if an *infinite* interval has an inside singularity.

has a singularity at 0. By Theorem 1 below,[3] we get

$$L\int_{-1}^{1/2}\frac{dx}{x^2}=\int_{-1}^{0-}\frac{dx}{x^2}+\int_{0+}^{1/2}\frac{dx}{x^2}$$
$$=\lim_{x\to0-}\Big(-\frac{1}{x}-1\Big)+\lim_{x\to0+}\Big(-2+\frac{1}{x}\Big)=\infty+\infty=\infty.$$

(B) We have

$$C\int_{1/2}^{\infty}\frac{dx}{x^2}=\lim_{x\to\infty}\Big(-\frac{1}{x}+2\Big)=2.$$

Hence

$$C\int_{-1}^{\infty}\frac{dx}{x^2}=C\int_{-1}^{1/2}\frac{dx}{x^2}+C\int_{1/2}^{\infty}\frac{dx}{x^2}=\infty+2=\infty.$$

(C) The integral

$$L\int_{-1}^{1}\frac{|x|}{x}dx$$

has *no* singularities (consider *deleted* globes about 0). The primitive $F(x)=|x|$ *exists* (example (b) in Chapter 5, §5); so

$$L\int_{-1}^{1}\frac{|x|}{x}dx=|x|\Big|_{-1}^{1}=0.$$

In the rest of this section, we state our theorems mainly for

$$C\int_{a}^{\infty}f,$$

but they apply, with similar proofs, to

$$C\int_{-\infty}^{\infty}f,\quad C\int_{a}^{b-}f,\quad \text{etc.}$$

The measure m is as explained above.

Theorem 1. *Let $A=[a,\infty)$, $f\colon E^1\to E$ (E complete).*

(i) *If $f\geq 0$ on A, then*

$$C\int_{a}^{\infty}f\,dm$$

exists ($\leq\infty$) and equals

$$\int_{A}f\,dm.^4$$

---

[3] It applies to *finite* intervals A, too.

[4] That is, the *proper* integral.

(ii) *The map f is m-integrable on A iff*

$$C\int_a^\infty |f| < \infty$$

and f is m-measurable on A; then again,

$$C\int_a^\infty f\,dm = \int_A f\,dm.$$

Proof. (i) Let $f \geq 0$ on A. By the rules of Chapter 8, §5, $\int_A f$ is always defined for such f; so we may set

$$F(x) = \int_a^x f\,dm, \quad x \geq a.$$

Then by Theorem 1(f) in Chapter 8, §5, $F\!\uparrow$ *on* A; for $a \leq x \leq y$ implies

$$F(x) = \int_a^x f \leq \int_a^y f = F(y).$$

Now, by the properties of monotone limits,

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \int_a^x f = C\int_a^\infty f$$

exists in E^*; so by Theorem 1 of Chapter 4, §2, it can be found by making x run over some *sequence* $x_k \to \infty$, say, $x_k = k$.

Thus set

$$A_k = [a, k], \quad k = 1, 2, \ldots.$$

Then $\{A_k\}\!\uparrow$ and

$$\bigcup A_k = A = [a, \infty),$$

i.e., $A_k \nearrow A$.

Moreover, by Note 4 in Chapter 8, §5, the set function $s = \int f$ is *σ-additive* and semifinite (≥ 0). Thus by Theorem 2 of Chapter 7, §4 (left continuity)

$$\int_A f\,dm = \lim_{k\to\infty} \int_{A_k} f = \lim_{k\to\infty} \int_a^k f = C\int_a^\infty f, \tag{1}$$

proving (i).

(ii) By clause (i),

$$C\int_a^\infty |f| = \int_A |f|\,dm$$

exists, as $|f| \geq 0$. Hence

$$C\int_a^\infty |f| < \infty$$

plus measurability amounts to *integrability* (Theorem 2 of Chapter 8, §6).

Moreover,

$$C\int_a^\infty |f| < \infty$$

implies the convergence of $C\int_a^\infty f$ (see Corollary 1 below). Thus as

$$\lim_{x\to\infty}\int_a^x f$$

exists, we proceed exactly as before (here $s = \int f$ is *finite*), proving (ii) also. ☐

Note 2. If $E \subseteq E^*$, *formula* (1) results even if f is not m-measurable.[5]

Note 3. While f cannot be *integrable* unless $|f|$ is (Corollary 2 of Chapter 8, §6), it can happen that

$$C\int f$$

converges even if

$$C\int |f| = \infty$$

(this is called *conditional* convergence). A case in point is

$$C\int_0^\infty \frac{\sin x}{x}\,dx;$$

see Problem 8.

Thus *C-integrals may be finite where proper integrals are* ∞ *or fail to exist* (a great advantage!). Yet they are deficient in other respects (see Problem 9(c)).

For our next theorem, we need the previously "starred" Theorem 2 in Chapter 4, §2. (Review it!) As we shall see, C-integrals resemble *infinite series*.

Theorem 2 (Cauchy criterion)**.** *Let* $A = [a, \infty)$, $f\colon E^1 \to E$, E *complete. Suppose*

$$\int_a^x f\,dm$$

exists for each $x \in A$. (*This is automatic if* $E \subseteq E^*$; *see Chapter* 8, §5.)

Then

$$C\int_a^\infty f$$

converges iff for every $\varepsilon > 0$, *there is* $b \in A$ *such that*

$$\left|\int_v^x f\,dm\right| < \varepsilon \quad \text{whenever } b \le v \le x < \infty,^6 \tag{2}$$

---

[5] This is true provided $\int_A f\,dm$ is finite or *orthodox*, so that $s = \int f$ is semifinite.

and

$$\left|\int_a^b f\,dm\right| < \infty. \tag{2'}$$

Proof. By additivity (Chapter 8, §5, Theorem 2; Chapter 8, §7, Theorem 3),

$$\int_a^x f = \int_a^v f + \int_v^x f$$

if $a \leq v \leq x < \infty$. (In case $E \subseteq E^*$, this holds even if f is not integrable; see Theorem 2, of Chapter 8, §5.)

Now, *if*

$$C\int_a^\infty f$$

converges, let

$$r = \lim_{x\to\infty} \int_a^x f\,dm \neq \pm\infty.$$

Then for any $\varepsilon > 0$, there is some

$$b \in [a, \infty) = A$$

such that

$$\left|\int_a^x f\,dm - r\right| < \frac{1}{2}\varepsilon \quad \text{for } x \geq b.$$

(Why may we use the *standard* metric here?)

Taking $x = b$, we get (2′). Also, if $a \leq b \leq v \leq x$, we have

$$\left|\int_a^x f\,dm - r\right| < \frac{1}{2}\varepsilon$$

and

$$\left|r - \int_a^\nu f\,dm\right| < \frac{1}{2}\varepsilon.$$

Hence by the triangle law, (2) follows also. Thus this b satisfies (2).

Conversely, suppose such a b exists for every given $\varepsilon > 0$. Fixing b, we thus have (2) and (2′). Now, with $A = [a, \infty)$, define $F\colon A \to E$ by

$$F(x) = \int_a^x f\,dm,$$

so

$$C\int_a^\infty f = \lim_{x\to\infty} F(x)$$

---

[6] Here and later, for LS integrals, replace $\int_v^x$ by $\int_{(v,x]}$ and $\int_b^x$ by $\int_{(b,x]}$.

if this limit exists. By (2),

$$|F(x)| = \left|\int_a^x f\,dm\right| \le \left|\int_a^b f\,dm\right| + \left|\int_b^x f\,dm\right| < \left|\int_a^b f\,dm\right| + \varepsilon$$

if $x \ge b$. Thus F is *finite* on $[b, \infty)$, and so we may again use the standard metric

$$\rho(F(x), F(v)) = |F(x) - F(v)| = \left|\int_a^x f\,dm - \int_a^v f\,dm\right| \le \left|\int_v^x f\,dm\right| < \varepsilon$$

if $x, v \ge b$. The existence of

$$C\int_a^\infty f\,dm = \lim_{x\to\infty} F(x) \ne \pm\infty$$

now follows by Theorem 2 of Chapter 4, §2. (We shall henceforth presuppose this "starred" theorem.)

Thus all is proved. □

Corollary 1. *Under the same assumptions as in Theorem 2, the convergence of*

$$C\int_a^\infty |f|\,dm$$

implies that of

$$C\int_a^\infty f\,dm.$$

Indeed,

$$\left|\int_v^x f\right| \le \int_v^x |f|$$

(Theorem 1(g) of Chapter 8, §5, and Problem 10 in Chapter 8, §7).

Note 4. We say that $C\int f$ converges *absolutely* iff $C\int |f|$ converges.

Corollary 2 (comparison test). *If $|f| \le |g|$ a.e. on $A = [a, \infty)$ for some $f, g\colon E^1 \to E$, then*

$$C\int_a^\infty |f| \le C\int_a^\infty |g|;$$

so the convergence of

$$C\int_a^\infty |g|$$

implies that of

$$C\int_a^\infty |f|.$$

For as $|f|, |g| \ge 0$, Theorem 1 reduces all to Theorem 1(c) of Chapter 8, §5.

Note 5. As we see, absolutely convergent C-integrals coincide with *proper* (finite) Lebesgue integrals of nonnegative or m-measurable maps. For *conditional* (i.e., nonabsolute) convergence, see Problems 6–9, 13, and 14.

Iterated C-Integrals. Let the product space $X \times Y$ of Chapter 8, §8 be

$$E^1 \times E^1 = E^2,$$

and let $p = m \times n$, where m and n are Lebesgue measure or LS measures in E^1. Let

$$A = [a, b],\ B = [c, d], \text{ and } D = A \times B.$$

Then the integral

$$\int_B \int_A f\,dm\,dn = \int_Y \int_X f\,C_D\,dm\,dn$$

is also written

$$\int_c^d \int_a^b f\,dm\,dn$$

or

$$\int_c^d \int_a^b f(x, y)\,dm(x)\,dn(y).$$

As usual, we write "dx" for "$dm(x)$" if m is *Lebesgue measure* in E^1; similarly for n.

We now define

$$\begin{aligned} C\int_a^\infty \int_c^\infty f\,dn\,dm &= \lim_{b\to\infty} \int_a^b \left(\lim_{d\to\infty} \int_c^d f(x, y)\,dn(y) \right) dm(x) \\ &= C\int_a^\infty \int_c^\infty f(x, y)\,dn(y)\,dm(x), \end{aligned} \tag{3}$$

provided the limits and integrals involved exist.

If the integral (3) is finite, we say that it *converges*. Again, convergence is *absolute* if it holds also with f replaced by $|f|$, and *conditional* otherwise. Similar definitions apply to

$$C\int_c^\infty \int_a^\infty f\,dm\,dn,\ C\int_{-\infty}^b \int_c^\infty f\,dn\,dm, \text{ etc.}$$

Theorem 3. *Let $f\colon E^2 \to E^*$ be p-measurable on E^2 (p, m, n as above). Then we have the following.*

(i*) *The Cauchy integrals*

$$C\int_{-\infty}^\infty \int_{-\infty}^\infty |f|\,dn\,dm \text{ and } C\int_{-\infty}^\infty \int_{-\infty}^\infty |f|\,dm\,dn$$

exist ($\leq \infty$), *and both equal*

$$\int_{E^2} |f|\, dp.$$

(ii*) *If one of these three integrals is finite, then*

$$C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\, dn\, dm \text{ and } C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\, dm\, dn$$

converge, and both equal

$$\int_{E^2} f\, dp.$$

(Similarly for $C\int_a^\infty \int_{-\infty}^b f\, dn\, dm$, etc.)

Proof. As m and n are σ-finite (finite on intervals!), f surely has σ-finite support.

As $|f| \geq 0$, clause (i*) easily follows from our present Theorem 1(i) and Theorem 3(i) of Chapter 8, §8.

Similarly, clause (ii*) follows from Theorem 3(ii) of the same section. ☐

Theorem 4 (passage to polars)**.** *Let $p =$ Lebesgue measure in E^2. Suppose $f\colon E^2 \to E^*$ is p-measurable on E^2. Set*

$$F(r, \theta) = f(r\cos\theta, r\sin\theta), \quad r > 0.$$

Then

(a) $C\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\, dx\, dy = C\int_0^\infty r\, dr \int_0^{2\pi} F\, d\theta$, *and*

(b) $C\displaystyle\int_0^{\infty}\int_0^{\infty} f\, dx\, dy = C\int_0^\infty r\, dr \int_0^{\pi/2} F\, d\theta$,

provided f is nonnegative or p-integrable on E^2 (for (a)*) or on* $(0,\infty)\times(0,\infty)$ *(for* (b)*).*[7]

Proof Outline. First let $f = C_D$, with D a "curved rectangle"

$$\{(r,\theta) \mid r_1 < r \leq r_2,\ \theta_1 < \theta \leq \theta_2\}$$

for some $r_1 < r_2$ in $X = (0,\infty)$ and $\theta_1 < \theta_2$ in $Y = [0, 2\pi)$. By elementary geometry (or calculus), the area

$$pD = \frac{1}{2}(r_2^2 - r_1^2)(\theta_2 - \theta_1)$$

(the difference between two circular sectors).

---

[7] Hence the integrals in (a) and (b) can also be treated as *proper* integrals.

For $f = C_D$, formulas (a) and (b) easily follow from

$$pD = L\int_{E^2} C_D\,dp.$$

(Verify!) Now, curved rectangles *behave like half-open intervals*

$$(r_1, r_2] \times (\theta_1, \theta_2]$$

in E^2, since Theorem 1 in Chapter 7, §1, and Lemma 2 of Chapter 7, §2, apply with the same proof. Thus they form a *semiring generating the Borel field in* E^2.

Hence show (as in Chapter 8, §8) that Theorem 4 holds for $f = C_D$ $(D \in \mathcal{B})$. Then take $D \in \mathcal{M}^*$. Next let f be elementary and nonnegative, and so on, as in Theorems 2 and 3 in Chapter 8, §8. ☐

Examples (continued)**.**

(D) Let

$$J = L\int_0^\infty e^{-x^2}\,dx;$$

so

$$\begin{aligned} J^2 &= \left(C\int_0^\infty e^{-x^2}\,dx\right)\left(C\int_0^\infty e^{-y^2}\,dy\right) \\ &= C\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy. \quad \text{(Why?)} \end{aligned}$$

Set

$$f(x, y) = e^{-(x^2+y^2)}$$

in Theorem 4(b). Then $F(r, \theta) = e^{-r^2}$; hence

$$\begin{aligned} J^2 &= C\int_0^\infty r\,dr\left(\int_0^{\frac{\pi}{2}} e^{-r^2}\,d\theta\right) \\ &= C\int_0^\infty re^{-r^2}\,dr \cdot \frac{\pi}{2} = -\frac{1}{4}\pi e^{-t}\Big|_0^\infty = \frac{1}{4}\pi. \end{aligned}$$

(Here we computed

$$\int re^{-r^2}\,dr$$

by substituting $r^2 = t$.) Thus

$$C\int_0^\infty e^{-x^2}dx = L\int_0^\infty e^{-x^2}dx = \sqrt{\frac{1}{4}\pi} = \frac{1}{2}\sqrt{\pi}. \tag{4}$$

Problems on Cauchy Integrals

1. Fill in all proof details in Theorems 1–3. Verify also at least some of the cases other than $\int_a^\infty f$. Check the validity for *LS-integrals* (footnote 6).

2. Prove Theorem 4 in detail.

2′. Verify Notes 2 and 3 and examples (A)–(D).

3. Assuming $a > 0$, verify the following:

(i) $\displaystyle\int_1^\infty \frac{1}{t}e^{-t}\,dt \le \int_1^\infty e^{-t}\,dt = \frac{1}{e}.$

[Hint: Use Corollary 2.]

(ii) $\displaystyle\int_1^\infty e^{-at}\,dt = \frac{e^{-a}}{a}.$

(iii) $\displaystyle\int_0^\infty e^{-at}\,dt = \frac{1}{a}.$

(iv) $\displaystyle\int_0^\infty e^{-at}\sin bt\,dt = \frac{b}{a^2+b^2}.$

4. Verify the following:

(i) $\displaystyle\int_1^\infty \int_1^\infty e^{-xy}\,dy\,dx = \int_1^\infty \frac{1}{x}e^{-x}\,dx \le \frac{1}{e}$ (*converges*, by 3(i)).

(ii) $\displaystyle\int_0^\infty \int_0^\infty e^{-xy}\,dy\,dx \ge \int_1^\infty \int_0^\infty e^{-xy}\,dy\,dx = \int_1^\infty \frac{1}{x}(1-e^{-x})\,dx \ge \int_1^\infty \left(\frac{1}{x} - e^{-x}\right)dx = \infty.$

Does this contradict formula (4) in the text, or Problem 5, which follows?

5. Let $f(x, y) = e^{-xy}$ and

$$g(x) = L\int_0^1 e^{-xy}\,dy;$$

so $g(0) = 1$. (Why?)

(i) Is g R-integrable on $A = [0, 1]$? Is f so on $A \times A$?

(ii) Find $g(x)$ using Corollary 1 in §1.

(iii) Find the value of

$$R\int_0^1 \int_0^1 e^{-xy}\,dy\,dx = R\int_0^1 g$$

to within 1/10.

[Hint: Reduce it to Problem 6(b) in §1.]

$\Rightarrow$**6.** Let $f, g\colon E^1 \to E^*$ be m-measurable on $A = [a, b)$, $b \leq \infty$. Prove the following:

(i) If

$$C\int_a^{b-} f^+ < \infty \text{ or } C\int_a^{b-} f^- < \infty,$$

then $C\int_a^{b-} f$ exists and equals

$$C\int_a^{b-} f^+ - C\int_a^{b-} f^- = \int_A f\,dm \text{ (proper).}$$

(ii) If $\int_a^{b-} f$ converges *conditionally* only, then

$$\int_a^{b-} f^+ = \int_a^{b-} f^- = +\infty.$$

(iii) In case $C\int_a^{b-} |f| < \infty$, we have

$$C\int_a^{b-} |f \pm g| = \infty$$

iff $C\int_a^{b-} |g| = \infty$; also,

$$C\int_a^{b-} (f \pm g) = C\int_a^{b-} f \pm C\int_a^{b-} g$$

if $C\int_a^{b-} g$ exists (finite or not).

$\Rightarrow$**7.** Suppose $f\colon E^1 \to E^*$ is m-integrable and sign-constant on each

$$A_n = [a_n, a_{n+1}), \quad n = 1, 2, \ldots,$$

but changes sign from A_n to A_{n+1}, with

$$\bigcup_{n=1}^{\infty} A_n = [a, \infty)$$

and $\{a_n\}\uparrow$ fixed.

Prove that if

$$\left|\int_{A_n} f\,dm\right| \searrow 0$$

as $n \to \infty$, then

$$C\int_a^{\infty} f$$

converges.

[Hint: Use Problem 10 in Chapter 4, §13.]

⇒8. Let

$$f(x) = \frac{\sin x}{x}, \quad f(0) = 1.$$

Prove that

$$C \int_0^\infty f(x)\, dx$$

converges *conditionally* only.

[Hints: Use Problem 7. Show that

$$C \int_0^\infty |f| = L \int_{(0,\infty)} |f| = L \int_0^\infty f^+ = L \int_0^\infty f^- = \infty.]$$

⇒9. (Additivity.) Given $f\colon E^1 \to E$ (E complete) and $a < b < c \leq \infty$, suppose that

$$\int_a^x f\, dm \neq \pm\infty$$

(*proper*) exists for each $x \in [a, c)$. Prove the following:

(a) $C \int_a^{b-} f$ and $C \int_{a+}^{b} f$ converge.

(b) If

$$C \int_b^{c-} f$$

converges, so does

$$C \int_a^{c-} f = C \int_a^{b-} f + C \int_b^{c-} f.$$

(c) *Countable* additivity does not necessarily hold for C-integrals.

[Hint: Use Problem 8 suitably splitting $[0, \infty)$.]

10. (Refined comparison test.) Given $f, g\colon E^1 \to E$ (E complete) and $b \leq \infty$, prove the following:

(i) If for some $a < b$ and $k \in E^1$,

$$|f| \leq |kg| \quad \text{on } [a, b)$$

then

$$\int_a^{b-} |g| < \infty \text{ implies } \int_a^{b-} |f| < \infty.$$

(ii) Such $a, k \in E^1$ do exist if

$$\lim_{t\to b-} \frac{|f(t)|}{|g(t)|} < \infty$$

exists.

(iii) If this limit is not zero, then

$$\int_a^{b-} |g| < \infty \textit{ iff } \int_a^{b-} |f| < \infty.$$

(Similarly in the case of $\int_{a+}^{b}$ with $a \geq -\infty$.)

11. Prove that

(i) $\displaystyle\int_1^{\infty} t^p \, dt < \infty$ iff $p < -1$;

(ii) $\displaystyle\int_{0+}^{1} t^p \, dt < \infty$ iff $p > -1$;

(iii) $\displaystyle\int_{0+}^{\infty} t^p \, dt = \infty$.

12. Use Problems 10 and 11 to test for convergence of the following:

(a) $\displaystyle\int_0^{\infty} \frac{t^{3/2}\, dt}{1+t^2}$;

(b) $\displaystyle\int_1^{\infty} \frac{dt}{t\sqrt{1+t^2}}$;

(c) $\displaystyle\int_a^{\infty} \frac{P(t)}{Q(t)}\, dt$
(Q, P polynomials of degree s and r, $s > r$; $Q \neq 0$ for $t \geq a$);

(d) $\displaystyle\int_0^{1-} \frac{dt}{\sqrt{1-t^4}}$;

(e) $\displaystyle\int_{0+}^{1} t^p \ln t \, dt$;

(f) $\displaystyle\int_0^{1-} \frac{dt}{\ln t}$;

(g) $\displaystyle\int_{0+}^{\frac{\pi}{2}-} \tan^p t \, dt$.

⇒**13.** (The Abel–Dirichlet test.) Given $f, g \colon E^1 \to E^1$, suppose that

(a) $f\downarrow$, with $\lim_{t \to \infty} f(t) = 0$;

(b) g is L-measurable on $A = [a, \infty)$;[8] and

(c) $(\exists\, K \in E^1)\ (\forall\, x \in A)\quad \left| L \int_a^x g \right| < K$.

---

[8] And hence *L-integrable* on each $[u, v] \subset A$, by (c).

Then $C\int_a^\infty f(x)\,g(x)\,dx$ *converges.*

[Outline: Set

$$G(x)=\int_a^x g;$$

so $|G|<K$ on A. By Lemma 2 of §1, fg is L-integrable on each $[u,v]\subset A$, and $(\exists\, c\in[u,v])$ such that

$$\left|L\int_u^v fg\right|=\left|f(u)\int_u^c g\right|=\bigl|f(u)\,[G(c)-G(u)]\bigr|<2Kf(u).$$

Now, by (a),

$$(\forall\,\varepsilon>0)\ (\exists\, k\in A)\ (\forall\, u\geq k)\quad |f(u)|<\frac{\varepsilon}{2K};$$

so

$$(\forall\, v\geq u\geq k)\quad \left|L\int_u^v fg\right|<\varepsilon.$$

Now use Theorem 2.

Now extend this to $g\colon E^1\to E^n\ (C^n)$.]

⇒14. Do Problem 13, replacing assumptions (a) and (c) by

(a′) *f is monotone and bounded on* $[a,\infty)=A$, and

(c′) $C\int_a^\infty g(x)\,dx$ *converges.*

[Hint: If $f\uparrow$, say, set $q=\lim_{t\to\infty} f(t)$ and $F=q-f$; so

$$fg=qg-Fg.$$

Apply Problem 13 to

$$C\int_a^\infty F(x)\,g(x)\,dx.]$$

15. Use Problems 13 and 14 to test the convergence of the following:

(a) $\displaystyle\int_0^\infty t^p\sin t\,dt.$

[Hint: The integral converges iff $p<0$.]

(b) $\displaystyle\int_{0+}^\infty \frac{\cos t}{\sqrt{t}}\,dt.$

[Hint: Integrate $\displaystyle\int_u^v \frac{\cos t}{\sqrt{t}}\,dt$ by parts; then let $u\to 0$ and $v\to\infty$.]

(c) $\displaystyle\int_1^\infty \frac{\cos t}{t^p}\,dt.$

(d) $\displaystyle\int_0^\infty \sin t^2\,dt.$

[Hint: Substitute $t^2=u$; then use (a).]

16. The *Cauchy principal value* (CPV) of $C\int_{-\infty}^{\infty} f(t)\,dt$ is defined by

$$(\mathrm{CPV})\int_{-\infty}^{\infty} f = \lim_{x\to\infty}\int_{-x}^{x} f(t)\,dt$$

(if it exists). Prove the following:

(i) If $C\int f(t)\,dt$ exists, so does $(\mathrm{CPV})\int f$, and the two are equal. Disprove the converse.
[Hint: Take $f(t) = \operatorname{sign}(t)/\sqrt{|t|}$.]

(ii) Do the same for

$$(\mathrm{CPV})\int_{a}^{b} f = \lim_{\delta\to 0+}\left(\int_{a}^{p-\delta} f + \int_{p+\delta}^{b} f\right),$$

p being the *only* singularity in (a, b).

§4. Convergence of Parametrized Integrals and Functions

I. We now consider C-integrals of the form

$$C\int f(t, u)\,dm(t),$$

where m is Lebesgue or LS measure in E^1. Here the variable u, called a *parameter*, remains *fixed* in the process of integration; but the end result depends on u, of course.

We assume $f\colon E^2 \to E$ (E *complete*) even if not stated explicitly. As before, we give our definitions and theorems for the case

$$C\int_{a}^{\infty}.$$

The other cases ($C\int_{-\infty}^{a}$, $C\int_{a}^{b-}$, etc.) are analogous; they are treated in Problems 2 and 3. We assume

$$a, b, c, x, t, u, v \in E^1$$

throughout, and write "dt" for "$dm(t)$" iff m is *Lebesgue* measure.

If

$$C\int_{a}^{\infty} f(t, u)\,dm(t)$$

converges *for each* u in a set $B \subseteq E^1$,[1] we can define a map $F\colon B \to E$ by

$$F(u) = C\int_a^\infty f(t,u)\,dm(t) = \lim_{x\to\infty}\int_a^x f(t,u)\,dm(t).$$

This means that

$$(1)\quad (\forall\, u \in B)\ (\forall\, \varepsilon > 0)\ (\exists\, b > a)\ (\forall\, x \geq b)\quad \left|\int_a^x f(t,u)\,dm(t) - F(u)\right| < \varepsilon,$$

so $|F| < \infty$ on B.

Here b depends on both ε and u (convergence is "*pointwise*"). However, it may occur that *one and the same* b fits *all* $u \in B$, so that b depends on ε alone. We then say that

$$C\int_a^\infty f(t,u)\,dm(t)$$

converges *uniformly* on B (i.e., for $u \in B$), and write

$$F(u) = C\int_a^\infty f(t,u)\,dm(t)\ (\textit{uniformly})\text{ on } B.$$

Explicitly, this means that

$$(2)\quad (\forall\, \varepsilon > 0)\ (\exists\, b > a)\ (\forall\, u \in B)\ (\forall\, x \geq b)\quad \left|\int_a^x f(t,u)\,dm(t) - F(u)\right| < \varepsilon.$$

Clearly, this implies (1), but not conversely. We now obtain the following.

Theorem 1 (Cauchy criterion). *Suppose*

$$\int_a^x f(t,u)\,dm(t)$$

exists for $x \geq a$ *and* $u \in B \subseteq E^1$. (*This is automatic if* $E \subseteq E^*$; *see Chapter* 8, §5.)

Then

$$C\int_a^\infty f(t,u)\,dm(t)$$

converges uniformly on B *iff for every* $\varepsilon > 0$, *there is* $b > a$ *such that*

$$(3)\qquad (\forall\, v, x \in [b,\infty))\ (\forall\, u \in B)\quad \left|\int_v^x f(t,u)\,dm(t)\right| < \varepsilon,^{2}$$

and

$$\left|\int_a^b f(t,u)\,dm(t)\right| < \infty.$$

---

[1] This statement shall *imply* that $\int_a^x f(t,u)\,dm(t) \neq \pm\infty$ exists for $x \geq a$, $u \in B$.

[2] For *LS*-integrals, replace $\int_v^x$ by $\int_{(x,v]}$ here and in the proof below.

Proof. The *necessity* of (3) follows as in Theorem 2 of §3. (Verify!)

To prove *sufficiency*, suppose the desired b exists for every $\varepsilon > 0$. Then for each (*fixed*) $u \in B$,

$$C \int_a^\infty f(t, u)\, dm(t)$$

satisfies Theorem 2 of §3. Hence

$$F(u) = \lim_{x \to \infty} \int_a^x f(t, u)\, dm(t) \neq \pm\infty \tag{4}$$

exists for every $u \in B$ (pointwise). Now, from (3), writing briefly $\int f$ for $\int f(t, u)\, dm(t)$, we obtain

$$\left| \int_v^x f \right| = \left| \int_a^x f - \int_a^v f \right| < \varepsilon$$

for all $u \in B$ and all $x > v \geq b$.

Making $x \to \infty$ (with u and v temporarily fixed), we have by (4) that

$$\left| F(u) - \int_a^v f \right| \leq \varepsilon \tag{5}$$

whenever $v \geq b$.

But by our assumption, b depends on ε *alone* (not on u). Thus unfixing u, we see that (5) establishes the *uniform* convergence of

$$\int_a^\infty f,$$

as required.[3] □

Corollary 1. *Under the assumptions of Theorem* 1,

$$C \int_a^\infty f(t, u)\, dm(t)$$

converges uniformly on B if

$$C \int_a^\infty |f(t, u)|\, dm(t)$$

does.

Indeed,

$$\left| \int_v^x f \right| \leq \int_v^x |f| < \varepsilon.$$

---

[3] Note that Theorem 1 essentially depends on the assumed *completeness* of E.

Corollary 2 (comparison test)**.** *Let $f\colon E^2 \to E$ and $M\colon E^2 \to E^*$ satisfy*

$$|f(t,u)| \le M(t,u)$$

for $u \in B \subseteq E^1$ and $t \ge a$.

Then

$$C\int_a^\infty |f(t,u)|\,dm(t)$$

converges uniformly on B if

$$C\int_a^\infty M(t,u)\,dm(t)$$

does.

Indeed, Theorem 1 applies, with

$$\left|\int_v^x f\right| \le \int_v^x M < \varepsilon.$$

Hence we have the following corollary.

Corollary 3 ("M-test")**.** *Let $f\colon E^2 \to E$ and $M\colon E^1 \to E^*$ satisfy*

$$|f(t,u)| \le M(t)$$

for $u \in B \subseteq E^1$ and $t \ge a$. Suppose

$$C\int_a^\infty M(t)\,dm(t)$$

converges. Then

$$C\int_a^\infty |f(t,u)|\,dm(t)$$

converges (uniformly) on B. So does

$$C\int_a^\infty f(t,u)\,dm(t)$$

by Corollary 1.

Proof. Set

$$h(t,u) = M(t) \ge |f(t,u)|.$$

Then Corollary 2 applies (with M replaced by h there). Indeed, the convergence of

$$C\int h = C\int M$$

is trivially "uniform" for $u \in B$, since M does not depend on u at all. □

Note 1. Observe also that, if $h(t,u)$ *does not depend* on u, then the (*pointwise*) and (*uniform*) convergence of $C\int h$ are trivially equivalent.

We also have the following result.

Corollary 4. *Suppose*

$$C\int_a^\infty f(t,u)\,dm(t)$$

converges (pointwise) on $B \subseteq E^1$. *Then this convergence is uniform iff*

$$\lim_{\nu\to\infty} C\int_v^\infty f(t,u)\,dm(t) = 0 \text{ (uniformly) on } B,$$

i.e., iff

$$(\forall\,\varepsilon>0)\ (\exists\,b>a)\ (\forall\,u\in B)\ (\forall\,v\geq b)\quad \left|C\int_v^\infty f(t,u)\,dm(t)\right| < \varepsilon.$$

The proof (based on Theorem 1) is left to the reader, along with that of the following corollary.

Corollary 5. *Suppose*

$$\int_a^b f(t,u)\,dm(t) \neq \pm\infty$$

exists for each $u \in B \subseteq E^1$.

Then

$$C\int_a^\infty f(t,u)\,dm(t)$$

converges (uniformly) on B *iff*

$$C\int_b^\infty f(t,u)\,dm(t)$$

does.

II. The Abel–Dirichlet tests for uniform convergence of series (Problems 9 and 11 in Chapter 4, §13) have various analogues for C-integrals. We give two of them, using the *second law of the mean* (Corollary 5 in §1).

First, however, we generalize our definitions, "unstarring" some ideas of Chapter 4, §11. Specifically, given

$$H\colon E^2 \to E \text{ (E complete)},$$

we say that $H(x,y)$ *converges to* $F(y)$, *uniformly on* B, *as* $x \to q$ ($q \in E^*$), and write

$$\lim_{x\to q} H(x,y) = F(y) \text{ (uniformly) on } B$$

iff we have

$$(6) \qquad (\forall\, \varepsilon > 0)\ (\exists\, G_{\neg q})\ (\forall\, y \in B)\ (\forall\, x \in G_{\neg q}) \quad |H(x, y) - F(y)| < \varepsilon;$$

hence $|F| < \infty$ on B.

If here $q = \infty$, the deleted globe $G_{\neg q}$ has the form (b, ∞). Thus if

$$H(x, u) = \int_a^x f(t, u)\, dt,$$

(6) turns into (2) as a special case. If (6) holds with "$(\exists\, G_{\neg q})$" and "$(\forall\, y \in B)$" *interchanged*, as in (1), convergence is *pointwise* only.

As in Chapter 8, §8, we denote by $f(\cdot\,, y)$, or f^y, the function *of x alone* (*on E^1*) given by

$$f^y(x) = f(x, y).$$

Similarly,

$$f_x(y) = f(x, y).$$

Of course, we may replace $f(x, y)$ by $f(t, u)$ or $H(t, u)$, etc.

We use *Lebesgue* measure in Theorems 2 and 3 below.

Theorem 2. *Assume $f, g\colon E^2 \to E^1$ satisfy*

(i) *$C \int_a^\infty g(t, u)\, dt$ converges (uniformly) on B;*

(ii) *each g^u $(u \in B)$ is L-measurable on $A = [a, \infty)$;*

(iii) *each f^u $(u \in B)$ is monotone ($\downarrow$ or $\uparrow$) on A;*[4] *and*

(iv) *$|f| < K \in E^1$* (bounded) *on $A \times B$.*

Then

$$C \int_a^\infty f(t, u)\, g(t, u)\, dt$$

converges uniformly on B.

Proof. Given $\varepsilon > 0$, use assumption (i) and Theorem 1 to choose $b > a$ so that

$$(7) \qquad \left| L \int_v^x g(t, u)\, dt \right| < \frac{\varepsilon}{2K},$$

written briefly as

$$\left| L \int_v^x g^u \right| < \frac{\varepsilon}{2K},$$

for all $u \in B$ and $x > v \geq b$, with K as in (iv).

---

[4] Briefly: "$f(t, u)$ is monotone *in* t, and $g(t, u)$ is measurable *in* t $(t \in A)$." It should be well noted that all f^u and g^u are functions *of* t on E^1.

Hence by (ii), each g^u $(u \in B)$ is *L-integrable* on any interval $[v, x] \subset A$, with $x > v \geq b$. Thus given such u and $[v, x]$, we can use (iii) and Corollary 5 from §1 to find that

$$L \int_v^x f^u g^u = f^u(v)\, L \int_v^c g^u + f^u(x)\, L \int_c^x g^u$$

for some $c \in [v, x]$.

Combining with (7) and using (iv), we easily obtain

$$\left| L \int_v^x f(t, u)\, g(t, u)\, dt \right| < \varepsilon$$

whenever $u \in B$ and $x > v \geq b$. (Verify!)

Our assertion now follows by Theorem 1. □

Theorem 3 (Abel–Dirichlet test). *Let* $f, g \colon E^2 \to E^*$ *satisfy*

(a) $\lim_{t \to \infty} f(t, u) = 0$ *(uniformly) for* $u \in B$;

(b) *each* f^u $(u \in B)$ *is nonincreasing* $(\downarrow)$ *on* $A = [0, \infty)$;

(c) *each* g^u $(u \in B)$ *is L-measurable on* A; *and*

(d) $(\exists\, K \in E^1)$ $(\forall\, x \in A)$ $(\forall\, u \in B)$ $\left|L \int_a^x g(t, u)\, dt\right| < K$.

Then

$$C \int_a^\infty f(t, u)\, g(t, u)\, dt$$

converges uniformly on B.

Proof Outline. Argue as in Problem 13 of §3, replacing Theorem 2 in §3 by Theorem 1 of the present section.

By Lemma 2 in §1, obtain

$$\left| L \int_v^x f^u g^u \right| = \left| f^u(v)\, L \int_a^x g^u \right| \leq K\, f(v, u)$$

for $u \in B$ and $x > v \geq a$.

Then use assumption (a) to fix k so that

$$|f(t, u)| < \frac{\varepsilon}{2K}$$

for $t > k$ and $u \in B$. □

Note 2. Via components, Theorems 2 and 3 extend to the case $g \colon E^2 \to E^n$ (C^n).

Note 3. While Corollaries 2 and 3 apply to *absolute* convergence only, Theorems 2 and 3 cover *conditional convergence*, *too* (a great advantage!). The theorems also apply if f or g is *independent* of u (see Note 1). This supersedes Problems 13 and 14 in §3.

Examples.

(A) The integral

$$\int_0^\infty \frac{\sin tu}{t}\,dt$$

converges uniformly on $B_\delta = [\delta, \infty)$ *if* $\delta > 0$, and pointwise on $B = [0, \infty)$.

Indeed, we can use Theorem 3, with

$$g(t, u) = \sin tu$$

and

$$f(t, u) = \frac{1}{t},\ f(0, u) = 1,$$

say. Then the limit

$$\lim_{t\to\infty} \frac{1}{t} = 0$$

is trivially uniform for $u \in B_\delta$, as f is *independent* of u. Thus assumption (a) is satisfied. So is (d) because

$$\left|\int_0^x \sin tu\,dt\right| = \left|\frac{1}{u}\int_0^{xu} \sin\theta\,d\theta\right| \le \frac{1}{\delta}\cdot 2.$$

(Explain!) The rest is easy.

Note that Theorem 2 fails here since assumption (i) is not satisfied.

(B) The integral

$$\int_0^\infty \frac{1}{t} e^{-tu} \sin at\,dt$$

converges uniformly on $B = [0, \infty)$. It does so *absolutely* on $B_\delta = [\delta, \infty)$, if $\delta > 0$.

Here we shall use Theorem 2 (though Theorem 3 works, too). Set

$$f(t, u) = e^{-tu}$$

and

$$g(t, u) = \frac{\sin at}{t},\ g(0, u) = a.$$

Then

$$\int_0^\infty g(t, u)\,dt$$

converges (substitute $x = at$ in Problem 8 or 15 in §3). Convergence is trivially uniform, by Note 1. Thus assumption (i) holds, and so do the other assumptions. Hence the result.

For *absolute* convergence on B_δ, use Corollary 3 with

$$M(t) = e^{-\delta t},$$

so $M \geq |fg|$.

Note that, quite similarly, one treats C-integrals of the form

$$\int_a^\infty e^{-tu} g(t)\,dt, \ \int_a^\infty e^{-t^2u} g(t)\,dt, \text{ etc.,}$$

provided

$$\int_a^\infty g(t)\,dt$$

converges ($a \geq 0$).

In fact, Theorem 2 states (roughly) that *the uniform convergence of* $C\int g$ *implies that of* $C\int fg$, provided f is monotone (in t) and bounded.

III. We conclude with some theorems on uniform convergence of *functions* $H\colon E^2 \to E$ (see (6)). In Theorem 4, m is again an LS (or Lebesgue) measure in E^1; the deleted globe $G^*_{\neg q}$ is fixed.

Theorem 4. *Suppose*

$$\lim_{x\to q} H(x,y) = F(y) \ (\textit{uniformly})^5$$

for $y \in B \subseteq E^1$. *Then we have the following*:

(i) *If all* H_x ($x \in G^*_{\neg q}$) *are continuous*[6] *or* m*-measurable on* B, *so also is* F.

(ii) *The same applies to* m-integrability *on* B, *provided* $mB < \infty$; *and then*

$$\lim_{x\to q} \int_B |H_x - F| = 0; \tag{8}$$

hence

$$\lim_{x\to q} \int_B H_x = \int_B F = \int_B \Big(\lim_{x\to q} H_x\Big). \tag{8'}$$

Formula (8′) is known as the *rule of passage to the limit under the integral sign.*

Proof. (i) Fix a sequence $x_k \to q$ (x_k in the deleted globe $G^*_{\neg q}$), and set

$$H_k = H_{x_k} \quad (k = 1, 2, \ldots).$$

The uniform convergence

$$H(x,y) \to F(y)$$

---

[5] *Pointwise* or *a.e.* convergence suffices for *m-measurability* in clause (i).

[6] Here and in Theorem 5, as functions *of* y: $H_x(y) = H(x,y)$. *Continuity may be relative or uniform.*

is preserved as x runs over that sequence (see Problem 4). Hence if all H_k are continuous or measurable, so is F (Theorem 2 in Chapter 4, §12 and Theorem 4 in Chapter 8, §1). Thus clause (i) is proved.

(ii) Now let all H_x be m-*integrable* on B; let

$$mB < \infty.$$

Then the H_k are m-measurable on B, and so is F, by (i). Also, by (6),

$$(\forall \varepsilon > 0)\ (\exists\, G_{\neg q})\ (\forall x \in G_{\neg q}) \quad \int_B |H_x - F| \le \int_B (\varepsilon) = \varepsilon\, mB < \infty,$$

proving (8). Moreover, as

$$\int_B |H_x - F| < \infty,$$

$H_x - F$ is m-*integrable* on B, and so is

$$F = H_x - (H_x - F).$$

Hence

$$\left|\int_B H_x - \int_B F\right| = \left|\int_B (H_x - F)\right| \le \int_B |H_x - F| \to 0,$$

as $x \to q$, by (8). Thus (8′) is proved, too. ☐

Quite similarly (keeping E complete and using *sequences*), we obtain the following result.

Theorem 5. *Suppose that*

(i) *all* H_x $(x \in G^*_{\neg q})$ *are continuous and finite on a finite interval* $B \subset E^1$, *and differentiable on* $B - Q$, *for a fixed countable set* Q;

(ii) $\lim_{x\to q} H(x, y_0) \ne \pm\infty$ *exists for some* $y_0 \in B$; *and*

(iii) $\lim_{x\to q} D_2 H(x, y) = f(y)$ (*uniformly*) *exists on* $B - Q$.

Then f, *so defined, has a primitive* F *on* B, exact on $B - Q$ (*so* $F' = f$ *on* $B - Q$); *moreover,*

$$F(y) = \lim_{x\to y} H(x, y)\ (\textit{uniformly})\ \textit{for } y \in B.$$

Outline of proof. Note that

$$D_2 H(x, y) = \frac{d}{dy} H_x(y).$$

Use Theorem 1 of Chapter 5, §9, with $F_n = H_{x_n}$, $x_n \to q$. ☐

Note 4. If $x \to q$ *over a path* P (clustering at q), one must replace $G_{\neg q}$ and $G^*_{\neg q}$ by $P \cap G_{\neg q}$ and $P \cap G^*_{\neg q}$ in (6) and in Theorems 4 and 5.

Problems on Uniform Convergence of Functions and C-Integrals

1. Fill in all proof details in Theorems 1–5, Corollaries 4 and 5, and examples (A) and (B).

1′. Using (6), prove that

$$\lim_{x \to q} H(x, y) \ (uniformly)$$

exists on $B \subseteq E^1$ iff

$$(\forall \varepsilon > 0)\ (\exists G_{\neg q})\ (\forall y \in B)\ (\forall x, x' \in G_{\neg q}) \quad |H(x, y) - H(x', y)| < \varepsilon.$$

Assume E complete and $|H| < \infty$ on $G_{\neg q} \times B$.

[Hint: "Imitate" the proof of Theorem 1, using *Theorem* 2 *of Chapter* 4, §2.]

2. State formulas analogous to (1) and (2) for $\int_{-\infty}^{a}$, $\int_{a}^{b-}$, and $\int_{a+}^{b}$.

3. State and prove Theorems 1 to 3 and Corollaries 1 to 3 for

$$\int_{-\infty}^{a}, \int_{a}^{b-}, \text{ and } \int_{a+}^{b}.$$

In Theorems 2 and 3 explore *absolute* convergence for

$$\int_{a}^{b-} \text{ and } \int_{a+}^{b}.$$

Do at least *some* of the cases involved.

[Hint: Use Theorem 1 of §3 and Problem 1′, if already solved.]

4. Prove that

$$\lim_{x \to q} H(x, y) = F(y) \ (uniformly)$$

on B iff

$$\lim_{n \to \infty} H(x_n, \cdot\,) = F \ (uniformly)$$

on B for all *sequences* $x_n \to q$ $(x_n \neq q)$.

[Hint: "Imitate" Theorem 1 in Chapter 4, §2. Use Definition 1 of Chapter 4, §12.]

5. Prove that if

$$\lim_{x \to q} H(x, y) = F(y) \ (uniformly)$$

on A and on B, then this convergence holds on $A \cup B$. Hence deduce similar propositions on *C-integrals*.

6. Show that the integrals listed below violate Corollary 4 and hence *do not* converge *uniformly* on $P = (0, \delta)$[7] though *proper* L-integrals exist for each $u \in P$. Thus show that Theorem 1(ii) does not apply to *uniform* convergence.

---

[7] Here and below, $\delta > 0$ is arbitrarily small.

(a) $\displaystyle\int_{0+}^{1} \frac{u\,dt}{t^2 - u^2}$;

(b) $\displaystyle\int_{0+}^{1} \frac{u^2 - t^2}{(t^2 + u^2)^2}\,dt$;

(c) $\displaystyle\int_{0+}^{1} \frac{tu(t^2 - u^2)}{(t^2 + u^2)^2}\,dt$.

[Hint for (b): To disprove uniform convergence, fix any $\varepsilon, v > 0$. Then

$$\int_0^v \frac{u^2 - t^2}{(t^2 + u^2)^2}\,dt = \frac{v}{v^2 + u^2} \longrightarrow \frac{1}{v}$$

as $u \to 0$. Thus if $v < \frac{1}{2\varepsilon}$,

$$(\exists\, u \in P) \quad \int_0^v \frac{u^2 - t^2}{(t^2 + u^2)^2}\,dt > \frac{1}{2v} > \varepsilon.]$$

7. Using Corollaries 3 to 5, show that the following integrals converge (*uniformly*) *on* U (as listed) but only pointwise *on* P (for the latter, proceed as in Problem 6). Specify P and $M(t)$ in each case where they are not given.

(a) $\displaystyle\int_0^\infty e^{-ut^2}\,dt$; $U = [\delta, \infty)$; $P = (0, \delta)$.
[Hint: Set $M(t) = e^{-\delta t}$ for $t \geq 1$ (Corollaries 3 and 5).]

(b) $\displaystyle\int_0^\infty e^{-ut}\, t^a \cos t\,dt$ $(a \geq 0)$; $U = [\delta, \infty)$.

(c) $\displaystyle\int_{0+}^{1} t^{u-1}\,dt$; $U = [\delta, \infty)$.

(c′) $\displaystyle\int_{0+}^{1} t^{u-1}(\ln t)^n\,dt$; $U = [\delta, \infty)$.

(d) $\displaystyle\int_{0+}^{1} t^{-u} \sin t\,dt$; $U = [0, \delta]$, $0 < \delta < 2$; $P = [\delta, 2)$; $M(t) = t^{1-\delta}$.
[Hint: Fix v so small that

$$(\forall\, t \in (0, v)) \quad \frac{\sin t}{t} > \frac{1}{2}.$$

Then, if $u \to 2$,

$$\int_0^v t^{-u} \sin t\,dt \geq \frac{1}{2}\int_0^v \frac{dt}{t^{u-1}} \to \infty.]$$

8. In example (A), disprove uniform convergence on $P = (0, \infty)$.
[Hint: Proceed as in Problem 6.]

9. Do example (B) using Theorem 3 and Corollary 5. Disprove uniform convergence *on* B.

10. Show that

$$\int_{0+}^{\infty} \frac{\sin tu}{t} \cos t \, dt$$

converges uniformly on any closed interval U, with $\pm 1 \notin U$.
[Hint: Transform into

$$\frac{1}{2} \int_{0+}^{\infty} \frac{1}{t} \big\{ \sin[(u+1)t] + \sin[(u-1)t] \big\} \, dt.]$$

11. Show that

$$\int_{0}^{\infty} t \, \sin t^3 \, \sin tu \, dt$$

converges (*uniformly*) on any finite interval U.
[Hint: Integrate

$$\int_{x}^{y} t \, \sin t^3 \, \sin tu \, dt$$

by parts *twice*. Then let $y \to \infty$ and $x \to 0$.]

12. Show that

$$\int_{0+}^{\infty} e^{-tu} \frac{\cos t}{t^a} \, dt \quad (0 < a < 1)$$

converges (*uniformly*) for $u \geq 0$.
[Hints: For $t \to 0+$, use $M(t) = t^{-a}$. For $t \to \infty$, use example (B) and Theorem 2.]

13. Prove that

$$\int_{0+}^{\infty} \frac{\cos tu}{t^a} \, dt \quad (0 < a < 1)$$

converges (*uniformly*) for $u \geq \delta > 0$, but (*pointwise*) for $u > 0$.
[Hint: Use Theorem 3 with $g(t, u) = \cos tu$ and

$$\left| \int_{0}^{x} g \right| = \left| \frac{\sin xu}{u} \right| \leq \frac{1}{\delta}.$$

For $u > 0$,

$$\int_{v}^{\infty} \frac{\cos tu}{t^a} \, dt = u^{a-1} \int_{vu}^{\infty} \frac{\cos z}{z} \, dz \to \infty$$

if $v = 1/u$ and $u \to 0$. Use Corollary 4.]

⇒**14.** Given $A, B \subseteq E^1$ $(mA < \infty)$ and $f \colon E^2 \to E$, suppose that

(i) each $f(x, \cdot\,) = f_x$ $(x \in A)$ is relatively (or uniformly) continuous on B; and

(ii) each $f(\,\cdot\,, y) = f^y$ $(y \in B)$ is m-integrable on A.

Set

$$F(y) = \int_A f(x, y)\, dm(x), \quad y \in B.$$

Then show that F is relatively (or uniformly) continuous on B.
[Hint: We have

$$(\forall\, x \in A)\ (\forall\, \varepsilon > 0)\ (\forall\, y_0 \in B)\ (\exists\, \delta > 0)\ (\forall\, y \in B \cap G_{y_0}(\delta))$$

$$|F(y) - F(y_0)| \leq \int_A |f(x, y) - f(x, y_0)|\, dm(x) \leq \int_A \Big(\frac{\varepsilon}{mA}\Big)\, dm = \varepsilon.$$

Similarly for *uniform* continuity.]

⇒**15.** Suppose that

(a) $C \int_a^\infty f(t, y)\, dm(t) = F(y)$ (*uniformly*) on $B = [b, d] \subseteq E^1$;

(b) each $f(x, \cdot\,) = f_x$ $(x \geq a)$ is relatively continuous on B; and

(c) each $f(\,\cdot\,, y) = f^y$ $(y \in B)$ is m-integrable on every $[a, x] \subset E^1$, $x \geq a$.

Then show that F is relatively continuous, hence integrable, on B and that

$$\int_B F = \lim_{x \to \infty} \int_B H_x,$$

where

$$H(x, y) = \int_a^x f(t, y)\, dm(t).$$

(*Passage to the limit under the* $\int$*-sign.*)
[Hint: Use Problem 14 and Theorem 4; note that

$$C \int_0^\infty f(t, y)\, dm(t) = \lim_{x \to \infty} H(x, y)\ (\textit{uniformly}).]$$

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